Exercise 1

Let $p$ and $q$ be two propositions. Show that

\[
B = \neg (p \rightarrow q) \rightarrow \neg (\neg p \rightarrow \neg q)
\]

is a tautology.

\[
B = \neg p \rightarrow \neg q
\]

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg q$</th>
<th>$p$</th>
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<th>$\neg p$</th>
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$B$ is always true: $B$ is a tautology.
Exercises 2

Let $p$ and $q$ be two propositions. I define the proposition $p$ NOR $q$ such that it is true when both $p$ and $q$ are false, and false otherwise. It is denoted $\neg p \lor \neg q$.

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<tr>
<th>$p$</th>
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<th>$p \lor q$</th>
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a) Show that $p \lor q$ is logically equivalent to $\neg (p \lor q)$. 
\[
\begin{array}{cccc}
  p & q & p \lor q & \neg(p \lor q) \\
  T & T & T & F \\
  T & F & T & F \\
  F & T & T & F \\
  F & F & F & T \\
\end{array}
\]

\[p \lor q \text{ and } \neg(p \lor q)\] always have the same truth values. They are logically equivalent.

\[p \lor q \iff \neg(p \lor q)\]

b) Find a compound proposition logically equivalent to \[p \lor q\] that only uses the logical operator \[\Rightarrow\].

De Morgan's law: \[
\neg(p \lor q) \iff \neg p \land \neg q
\]
\[ p \rightarrow q \iff \neg(p \lor q) \]
\[ \iff \neg p \land \neg q \]

Rewrite this as:

\[ \neg p \lor \neg q \iff \neg(p \land q) \]

\[ A \iff \neg p \]
\[ B \iff \neg q \]

\[ A \land B \iff \neg A \lor \neg B \]

\[ p \lor q \iff \neg (\neg p \lor q) \]
\[ \iff \neg (p \lor q) \]
\[ \iff \neg q = p \]
\[ p \lor p \iff \neg (p \lor p) \]
\[ \iff \neg p \]

\[ A \land B \iff (A \downarrow A) \downarrow (B \downarrow B) \]
$$A \land B \iff (A \lor A) \lor (B \lor B)$$

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have the same truth values; they are logically equivalent.

c) Find a compound proposition that is logically equivalent to $p \implies q$ that only uses the logical operator $\lor$.

\[
\begin{align*}
  p \lor q & \iff \neg (p \lor q) \\
  \neg p & \iff p \lor p \\
  p \lor q & \iff (p \lor p) \lor (q \lor q)
\end{align*}
\]
\[ p \to q \iff \neg p \lor q \]

De Morgan's Law:
\[ \neg (A \land B) \iff \neg A \lor \neg B \]

\[ \neg (p \land \neg q) \iff \neg p \lor q \]

\[ p \to q \iff \neg (p \land \neg q) \]
\[ \iff \neg (p \land (q \lor \neg q)) \]
\[ \iff \neg (p \land \top) \iff \neg (p \lor (q \land \neg q)) \]

\[ p \to q \iff \neg (p \lor q \lor (q \land \neg q)) \iff \neg (p \lor \bot) \]

\[ p \to q \iff \neg (p \lor q) \]

\[ p \to q \iff \neg (p \lor q) \]

\[ p \to q \iff \neg (p \lor q) \]

\[ p \to q \iff \neg (p \lor q) \]

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\[ p \to q \iff \neg (p \lor q) \]

\[ p \to q \iff \neg (p \lor q) \]

\[ p \to q \iff \neg (p \lor q) \]

\[ p \to q \iff \neg (p \lor q) \]
You find yourself in front of 3 rooms with closed doors. You are told that behind one door there is a princess, and behind the two others doors there are tigers. The rooms are guarded by guardians. Those guardians either tell the truth or lie.

| Room | Guardian
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>Knight</td>
</tr>
<tr>
<td>2</td>
<td>Knave</td>
</tr>
<tr>
<td>3</td>
<td>Knave</td>
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</table>

- **Room 1**: The princess is behind my door.
- **Room 2**: There is exactly one liar among us, and the princess is behind my door.
- **Room 3**: We are all liars.

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<th>S₁</th>
<th>S₂</th>
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<td>Knave, Knave, Knight</td>
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<tr>
<td>Knave, Knave, Knave</td>
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