Exercise 1

Let $n$ be an integer. Show that if $2n^2 + n + 9$ is odd, then $n$ is even using an indirect proof, a proof by contradiction, and a direct proof.

This is a problem of showing a conditional $p \rightarrow q$ is true, where

$p : 2n^2 + n + 9$ is odd
$q : n$ is even

We will use three different types of proof: indirect, proof by contradiction, and direct proof:

a) Indirect proof: we show that $\neg q \rightarrow \neg p$ is true

Hypothesis: $\neg q$ is true, namely $n$ is odd.

Since $n$ is odd, there exists an integer $k$ such that $n = 2k + 1$. Therefore, $2n^2 + n + 9 = 2(2k + 1)^2 + (2k + 1) + 9 = 8k^2 + 10k + 12 = 2(4k^2 + 5k + 6)$

Since $4k^2 + 5k + 6$ is integer, $2n^2 + n + 9$ is even, therefore $\neg p$ is true. Therefore $\neg q \rightarrow \neg p$ is true, and $p \rightarrow q$ is true.

b) Proof by contradiction: we suppose $p \rightarrow q$ is false

Hypothesis: $p \rightarrow q$ is false, i.e. $p$ is true AND $\neg q$ is true, namely $2n^2 + n + 9$ is odd and $n$ is odd.

Since $n$ is odd, there exists an integer $k$ such that $n = 2k + 1$. Therefore, $2n^2 + n + 9 = 2(2k + 1)^2 + (2k + 1) + 9 = 8k^2 + 10k + 12 = 2(4k^2 + 5k + 6)$

Since $4k^2 + 5k + 6$ is integer, $2n^2 + n + 9$ is even. But we have supposed that $2n^2 + n + 9$ is odd. We have reached a contradiction. Therefore the hypothesis we made is false, therefore $p \rightarrow q$ is true.

c) Direct proof: we show directly that $p \rightarrow q$ is true.

Hypothesis: $p$ is true, $2n^2 + n + 9$ is odd. Therefore there exists an integer $k$ such that $2n^2 + n + 9 = 2k + 1$, i.e. $n = 2k - 2n^2 - 8 = 2(k - n^2 - 4)$. Since $k - n^2 - 4$ is an integer, we conclude that 2 divides $n$, therefore $n$ is even. We have showed that $q$ is true, therefore $p \rightarrow q$ is true.
Exercise 2

Let $p$ be a natural number. Show that $2^{\frac{1}{p}}$ is irrational.

We use a proof by contradiction: let us suppose that $2^{\frac{1}{4}}$ is a rational number. There exists two integers $a$ and $b$, with $b \neq 0$ such that

\[ 2^{\frac{1}{4}} = \frac{a}{b} \]

(1)

After raising this equation to the power 2, we get:

\[ \sqrt{2} = \frac{a^2}{b^2} \]

(2)

As $a$ and $b$ are integers; $a^2$ and $b^2$ are integers, with $b^2 \neq 0$. The equation above would then mean that $\sqrt{2}$ is rational; this is not true. Therefore $2^{\frac{1}{4}}$ is irrational.

Exercise 3

Let $a$ and $b$ be two integers. Show that if either $ab$ or $a + b$ is odd, then either $a$ or $b$ is odd.

This is an implication of the form $p \rightarrow q$, with:

- $p$: $ab$ is odd or $a + b$ is odd
- $q$: $a$ is odd or $b$ is odd

where $a$ and $b$ are integers.

We use an indirect proof (proof by contrapositive).

Hypothesis: $\neg q$: $a$ is even and $b$ is even.

There exist two integers $k$ and $l$ such that $a = 2k$ and $b = 2l$. Then $ab = 2k \times 2l = 4kl = 2(2kl)$ therefore there exists an integer $m(= 2kl)$ such that $ab = 2m$: $ab$ is even.

and $a + b = 2k + 2l = 2(k + l)$ therefore there exists an integer $n(= k + l)$ such that $a + b = 2n$: $a + b$ is even.

We have proved that $ab$ is even and $a + b$ is even; $\neg p$ is true. Therefore $\neg q \rightarrow \neg p$ is true, and by contrapositive, $p \rightarrow q$ is true.

Exercise 4

Let $a$ and $b$ be two integers. Show that if $a^2(b^2 - 2b)$ is odd, then $a$ is odd and $b$ is odd.

This is an implication of the form $p \rightarrow q$, with:

- $p$: $a^2(b^2 - 2b)$ is odd
- $q$: $a$ is odd and $b$ is odd

where $a$ and $b$ are integers.

We use an indirect proof (proof by contrapositive).

Hypothesis: $\neg q$: $a$ is even or $b$ is even. We look at both cases:
Case 1: $a$ is even.

There exists an integer $k$ such that $a = 2k$. Then $a^2(b^2 - 2b) = 4k^2(b^2 - 2b) = 2[2k^2(b^2 - 2b)]$. Since $2k^2(b^2 - 2b)$ is an integer, we conclude that $a^2(b^2 - 2b)$ is even.

Case 2: $b$ is even.

There exists an integer $l$ such that $b = 2l$. Then $a^2(b^2 - 2b) = a^2(4l^2 - 4l) = 2[a^2(2l^2 - 2l)]$. Since $a^2(2l^2 - 2l)$ is an integer, we conclude that $a^2(b^2 - 2b)$ is even.

In both cases we have shown that $a^2(b^2 - 2b)$ is even, i.e. that $\lnot p$ is true. Therefore $\lnot q \rightarrow \lnot p$ is true, and by contrapositive, $p \rightarrow q$ is true.