Induction

Exercise 1

Show that

\[ \forall n \in \mathbb{N}, \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n + 1)}{2} \]

Let \( P(n) \) be the predicate:

\[ P(n) : \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n + 1)}{2} \]

We want to show that \( P(n) \) is true for all natural numbers.

Let us define: \( LHS(n) = \sum_{i=1}^{n} (-1)^i i^2 \) and \( RHS(n) = \frac{(-1)^n n(n + 1)}{2} \). \( P(n) \) is then \( LHS(n) = RHS(n) \). To prove that \( P(n) \) is true for all \( n \geq 1 \), we use a proof by induction.

- **Basis step:**
  \[ LHS(1) = (-1) \times 1^2 = 1 \quad \text{and} \quad RHS(1) = \frac{(-1) \times 1 \times 2}{2} = 1 \]
  Therefore \( P(1) \) is true.

- **Induction step:** We suppose that \( P(n) \) is true, with \( 1 \leq n \). We want to show that \( P(n + 1) \) is true.
\[
LHS(n+1) = \sum_{i=1}^{n+1} (-1)^i i^2 \\
= \sum_{i=1}^{n} (-1)^i i^2 + (-1)^{n+1} (n+1)^2 \\
= LHS(n) + (-1)^{n+1} (n+1)^2 \\
= RHS(n) + (-1)^{n+1} (n+1)^2 \\
= -\frac{1}{2} n(n+1) + (-1)^{n+1} (n+1)^2 \\
= -\frac{1}{2} n(n+1)(2n+2-2n) \\
= -\frac{1}{2} n(n+1)(n+2)
\]

and

\[
RHS(n+1) = -\frac{1}{2} (-1)^{n+1} (n+1)(n+2)
\]

Therefore \(LHS(n+1) = RHS(n+1)\), which validates that \(P(n+1)\) is true.

The principle of proof by mathematical induction allows us to conclude that \(P(n)\) is true for all \(n > 0\).

**Exercise 2**

*Show that*

\[
\forall n \geq 4, \quad 2^n \leq n!
\]

Let us define \(LHS(n) = 2^n\) and \(RHS(n) = n!\). Let \(P(n)\) be the proposition: \(LHS(n) \leq RHS(n)\). We want to show that \(P(n)\) is true for all \(n \geq 4\).

- **Basis step**: We show that \(P(4)\) is true:

  \[
  LHS(4) = 2^4 = 16 \\
  RHS(4) = 4! = 24
  \]

  Therefore \(LHS(4) \leq RHS(4)\) and \(P(4)\) is true.

- **Inductive step**: Let \(n\) be a positive integer greater or equal to 4 \((n \geq 4)\), and let us suppose that \(P(n)\) is true. We want to show that \(P(n+1)\) is true.

  \[
  LHS(n+1) = 2^{n+1} = 2LHS(n)
  \]
Since $P(n)$ is true, we find:

$$LHS(n + 1) \leq 2n!$$

Since $n \geq 4$, $2 \leq n + 1$.
Therefore

$$LHS(n + 1) \leq (n + 1) \times n!$$
$$LHS(n + 1) \leq (n + 1)!$$

Since $RHS(n + 1) = (n + 1)!$, we get $LHS(n + 1) < RHS(n + 1)$ which validates that $P(n + 1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 4$.

**Exercise 3**

*Show that*

$$\forall n \geq 1, \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Let us define: $LHS(n) = \sum_{i=1}^{n} \frac{1}{(i)(i+1)}$ and $RHS(n) = \frac{n}{n+1}$. Let $P(n)$ be the proposition: $LHS(n) = RHS(n)$. We want to show that $P(n)$ is true for all $n > 0$. We use a proof by induction.

- **Basis step:**

  $$LHS(1) = \frac{1}{1 \times 2} = \frac{1}{2} \quad RHS(1) = \frac{1}{2}$$

  Therefore $LHS(1) = RHS(1)$, and $P(1)$ is true.

- **Induction step:** We suppose that $P(n)$ is true, with $1 \leq n$. We want to show that $P(n + 1)$ is true.
\[
LHS(n + 1) = \sum_{i=1}^{n+1} \frac{1}{i(i+1)} \\
= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\
= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
= \frac{n(n+2) + 1}{(n+1)(n+2)} \\
= \frac{n^2 + 2n + 1}{(n+1)(n+2)} \\
= \frac{(n+1)^2}{(n+1)(n+2)} \\
= \frac{n+1}{n+2}
\]

and

\[
RHS(n + 1) = \frac{n + 1}{n + 2}
\]

Therefore \(LHS(n + 1) = RHS(n + 1)\), which validates that \(P(n + 1)\) is true.

The principle of proof by mathematical induction allows us to conclude that \(P(n)\) is true for all \(n\).