Discussion 2/14

1. Let \( n \) be an integer. Show that \( n \) is even if and only if \( 9n + 16 \) is even.

- \( p: n \) is even
- \( \neg p: n \) is odd
- \( q: 9n + 16 \) is even
- \( \neg q: 9n + 16 \) is odd

I need to show \( p \Rightarrow q \) and \( q \Rightarrow \neg p \).

a) Let us start with \( p \Rightarrow q \).

*Direct proof.* I assume \( p \) is true.

There exists \( k \in \mathbb{Z} \) such that \( n = 2k \)

\[
9n + 16 = 9(2k) + 16 = 2(9k + 8)
\]

Since \( 9k + 8 \) is an integer, \( 9n + 16 \) is even.

Therefore \( q \) is true. I have shown \( p \Rightarrow q \) is true.
b) I need to show \( q \rightarrow p \)

Indirect proof: I start by assuming that \( \neg p \) is true. 

\( n \) is odd. There exists an integer such that \( n = 2k + 1 \)

\[
9n + 16 = 9(2k + 1) + 16 \\
= 18k + 9 + 16 \\
= 18k + 25 \\
= 2(9k + 12) + 1
\]

\( 9k + 12 \) is an integer. Therefore, \( 9n + 16 \) is odd.

I have shown \( \neg q \) is true, therefore \( \neg p \rightarrow \neg q \), \( q \rightarrow p \).

Direct proof: I assume \( q \) is true.

\( 9n + 16 \) is even: there exists an integer \( k \) such that

\[
9n + 16 = 2k \\
9n = 2k - 16 = 2(k - 8)
\]

This means \( 9n \) is even, since \( k - 8 \) is an integer.

We want to show that, given an integer \( n \),

if \( 9n \) is even, then \( n \) is even.

\[
9m = 2(k - 8) \\
m + 8m = 2(k - 8) \\
m = 2(k - 8 - 4m)
\]
Let $a$ and $b$ be two strictly positive real numbers.

Show that \( \frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}} \).
What we cannot do:

Assume picture \( \left( \frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}} \right) \)

\[
\frac{ab}{a} + \frac{ab}{b} \geq \frac{2ab}{\sqrt{ab}}
\]

\[
b + a \geq 2\sqrt{ab}
\]

\[
b + a - 2\sqrt{ab} \geq 0
\]

\[
\left( \sqrt{b} - \sqrt{a} \right)^2 \geq 0
\]

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Direct proof: \( a \) and \( b \) are strictly positive.

Let \( A = \left( \sqrt{b} - \sqrt{a} \right)^2 \)

Since \( A \) is a square,

\[
A \geq 0
\]

\[
\left( \sqrt{b} - \sqrt{a} \right)^2 \geq 0
\]

\[
b + a - 2\sqrt{ab} \geq 0
\]

Observe

\[
\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}
\]
Proof by contradiction.

We assume \(-p\) is true.

We assume
\[
\frac{1}{a} + \frac{1}{b} < \frac{2}{\sqrt{ab}}
\]
\[
b + a - 2\sqrt{ab} < 0
\]
\[
(\sqrt{b} - \sqrt{a})^2 < 0
\]

However, a square is always positive on \(0\), therefore I have reached a contradiction. My hypothesis was wrong: \(p\) is true.

Let \(a\) and \(b\) be two strictly positive real numbers. Show that \(p: \frac{a+b}{2} \geq \sqrt{ab}\).

Proof by contradiction. Assume \(p\) is false.

\[
\frac{a+b}{2} < \sqrt{ab}
\]
\[
a + b - 2\sqrt{ab} < 0
\]
\[
(a - \sqrt{b})^2 < 0
\]

Contradiction.