Final: solutions

Part 1: Data (10 questions, each 3 points; total 30 points)

1) 1) What is the largest unsigned integer that can be stored on 2 bytes?
   a) 256
   b) 255
   c) 65535
   d) 65536

   2 bytes = 16 bits; largest unsigned integer is $2^{16} - 1 = 65535$

2) 2) Convert the binary number (1101101111110101)\_2 to hexadecimal
   a) #DBF5
   b) #DCF5
   c) #5FBD
   d) #5FC5

3) (1110)\_2 - (101)\_2 =
   a) #B
   b) #8
   c) #A
   d) #9

   Note that:
   a) (1110)\_2 = (14)\_10
   b) (101)\_2 = (5)\_10

   Therefore (1110)\_2 - (101)\_2 = (9)\_10 = #9.
4) Which of these sampling rates would be appropriate for a sound sample of maximum frequency 16 kHz (circle all that apply)?

a) 16000 Hz
b) 8000 Hz
c) 35000 Hz
d) 35 Hz

Samples need to be collected at a rate strictly larger that $2 \times 16000$ Hz, i.e. at a rate strictly larger that 32000 Hz.

5) Assume that you have taken a square picture with a 4 megapixel digital camera. Assume that you are printing this picture out on a printer that has approximately 4000 dots per inch. How big would the picture be? (note: 1 dot = 1 pixel)

a) 1 inch $\times$ 1inch
b) 2 inches $\times$ 2 inches
c) 0.5 inch $\times$ 0.5 inch
d) 4 inches $\times$ 4 inches

The picture include 4,000,000 pixels; as it is square, it has 2000 pixels along both dimensions. 2000 pixels corresponds to 0.5 inch.

6) Which binary number comes right after the binary number 101?

a) 1110
b) 111
b) 110

7) Decode the name whose ASCII representation is #53 #61 #6C #6C #79

a) Sally
b) Sylla
c) SALLY
d) SYLLA

8) The highest frequency note for a piano is $f_c=4200$ Hz. Assuming that you record 1 hour of piano music with a sampling rate 3 times $f_c$, in mono, with 16 bits resolution, what is the size of the resulting file (assuming 1MB = 1,000,000 bytes):  

a) 0.9072 MB
b) 90.72 MB
c) 9.072 MB
d) 181.44 MB
Storage: $1 \text{ (hour)} \times 3600 \text{ (second/hour)} \times 3 \times 4200 \text{ (samples)} \times 16 \text{ (bits)} \times 1 \text{ (mono)} = 725760000 \text{ (bits)} = 90720000 \text{ bytes} = 90.72 \text{ megabytes}.$

Total: $1800+720=2520 \text{ megabytes}.$

9) **What time is it on this digital clock (filled circle mean on)?**

- a) 10:37
- b) 10:41
- c) 18:37
- d) 18:41

10) **You want to store an electronic copy of a book on your computer. This book contains 500 pages; each page contains (on average) 60 lines, and each line contains 60 characters (again, on average), including space. Each character needs 2 bytes of storage. How much space do you need to store this book (assuming 1MB = 1,000,000 bytes)?**

- a) 3.6 MB
- b) 36 MB
- c) 0.36 MB
- d) 360 MB

Size $= 500 \text{ (pages)} \times 60 \text{ (lines)} \times 60 \text{ (characters)} \times 2 \text{ (bytes)} = 3600000 \text{ (bytes)} = 3.6 \text{ MB.}$
1) For each of the five propositions in the table below, indicates on the right if they are always tautologies or not (p and q are propositions). (10 points)

<table>
<thead>
<tr>
<th>Proposition</th>
<th>Tautology (Yes/ No)</th>
</tr>
</thead>
<tbody>
<tr>
<td>If 2+6=5, then 10=-9</td>
<td>Yes: This is ( p \rightarrow q ) where ( p ) is false: therefore ( p \rightarrow q ) is true</td>
</tr>
<tr>
<td>((p \lor -p) \rightarrow q)</td>
<td>No: ( p \lor -p ) is always true, but the implication is true only if ( q ) is true.</td>
</tr>
<tr>
<td>((p \land -q) \rightarrow p)</td>
<td>Yes: ((p \land -q) \rightarrow p \equiv (\neg p \lor q \lor p) \equiv T \lor q \equiv T)</td>
</tr>
<tr>
<td>if 3+3=6 then 25=16+9</td>
<td>Yes! This is ( p \rightarrow q ) where ( p ) and ( q ) are true: therefore ( p \rightarrow q ) is true</td>
</tr>
<tr>
<td>((p \land -p) \lor (-p \lor p))</td>
<td>Yes! ((p \land -p) \lor (-p \lor p) \equiv F \lor T \equiv T)</td>
</tr>
</tbody>
</table>

2) A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You go to this island, as you have been told that a treasure may be buried on it. You meet two inhabitants, John, and Sally. John tells you that, “I am a knight if and only if the treasure is on the island.” Sally tells you that “If John is a knight, then the treasure is not on the island.” Can it be determined if the treasure is on the island? Can it be determined also whether John is a knight or knave? What about Sally? Justify your answers. (10 points)

John and Sally can be a knight or a knave, and the treasure is either present on the island or not.

<table>
<thead>
<tr>
<th>Line number</th>
<th>John</th>
<th>Sally</th>
<th>Treasure on island</th>
<th>John says</th>
<th>Sally says</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Knight</td>
<td>Knight</td>
<td>Yes</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>Knight</td>
<td>Knight</td>
<td>No</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>3</td>
<td>Knight</td>
<td>Knave</td>
<td>Yes</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>Knight</td>
<td>Knave</td>
<td>No</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>5</td>
<td>Knave</td>
<td>Knight</td>
<td>Yes</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>6</td>
<td>Knave</td>
<td>Knight</td>
<td>No</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>7</td>
<td>Knave</td>
<td>Knave</td>
<td>Yes</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>8</td>
<td>Knave</td>
<td>Knave</td>
<td>No</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

We can eliminate:
- Line 1, as Sally would be a knight but she lies
- Line 2, as John would be a knight but he lies
- Line 4, as John would be a knight but he lies
- Line 6, as John would be a knave but he tells the truth
- Line 7, as Sally would be a knave but she tells the truth
- Line 8, as John would be a knave but he tells the truth

The only options are Line 3 and 5; The treasure is on the island, but we only know that John and Sally are of opposite types.

3) Let \( p \) and \( q \) be two propositions. Use a truth table or logical equivalence to indicate if the proposition \( \neg(p \rightarrow \neg q) \rightarrow \neg(p \leftrightarrow \neg q) \) is a tautology, a contradiction, or neither

Let us build the truth table for \( A = \neg(p \rightarrow \neg q) \rightarrow \neg(p \leftrightarrow \neg q) \):

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg q )</th>
<th>( (p \rightarrow \neg q) )</th>
<th>( \neg(p \rightarrow \neg q) )</th>
<th>( p \leftrightarrow \neg q )</th>
<th>( \neg(p \leftrightarrow \neg q) )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

From Column 8, we can conclude that \( A \) is a tautology.

Part III 4 problems, each 10 points; total 40 points)

1) Give a direct proof and a proof by contradiction of the proposition: if \( 2n^3 + 3n^2 + 4n + 3 \) is odd, then \( n \) is even, where \( n \) is a natural number.

Let \( p \) be the proposition “\( 2n^3 + 3n^2 + 4n + 3 \) is odd” and \( q \) be the proposition “\( n \) is even”. We want to show that \( p \rightarrow q \) is true.

i) Let us show \( p \rightarrow q \) using a direct proof:

Hypothesis: \( p \) is true, i.e. \( 2n^3 + 3n^2 + 4n + 3 \) is odd. There exists an integer \( k \) such that let \( 2n^3 + 3n^2 + 4n + 3 = 2k + 1 \), We get:

\[
\begin{align*}
n^2 &= -2n^3 - 2n^2 - 4n + 2k - 2 \\
&= 2(-n^3 - n^2 - 2n + k - 1)
\end{align*}
\]

As \( -n^3 - n^2 - 2n + k - 1 \) is an integer, \( n^2 \) is even. We know that if \( n^2 \) is even, then \( n \) is even. Therefore \( n \) is even. We have shown that \( q \) is true when \( p \) is true: \( p \rightarrow q \) is true.

ii) Let us show \( p \rightarrow q \) using a proof by contradiction:

We use a proof by contradiction, i.e. we assume that \( p \) is true AND \( \neg q \) is true. Since \( \neg q \) is true, \( n \) is odd. If \( n \) is odd, then there exists an integer \( k \) such \( n = 2k + 1 \). We get:
\[2n^3 + 3n^2 + 4n + 3 = 2(2k + 1)^3 + 3(2k + 1)^2 + 4(2k + 1) + 3 = 2(8k^3 + 12k^2 + 6k + 1) + 3(4k^2 + 4k + 1) + 8k + 7 = 16k^3 + 36k^2 + 32k + 12 = 2(8k^3 + 18k^2 + 16k + 6)\]

As \(8k^3 + 18k^2 + 16k + 6\) is an integer, \(2n^3 + 3n^2 + 4n + 3\) is even. Remember however that we had assumed that \(p\) is true, i.e. that \(2n^3 + 3n^2 + 4n + 3\) is odd. We have reached a contradiction, and therefore \(p \rightarrow q\) is true.

2) **Use induction to prove that any postage value of \(n\) cents can be made with only 5-cent stamps and 6-cent stamps, whenever \(n \geq 20\), \(n\) natural number.**

Let \(p(n)\) be the proposition that \(n\) cents can be made with only 5-cent and 6-cent stamps, when \(n\) is greater than or equal to 20.

Therefore there exists two positive integers \(a_n\) and \(b_n\) such that \(n = 5a_n + 6b_n\)

a) Basis step: I want to prove that \(p(20)\) is true

\(20\) can be composed of 5 times 4 plus 0 times 6: \(20 = 5 \times 4 + 6 \times 0\)

We can set \(a_{20} = 4\) and \(b_{20} = 0\). Both are positive integers. Therefore \(p(20)\) is true

b) Inductive Step

I want to show \(p(n) \rightarrow p(n + 1)\) whenever \(n \geq 20\)

Hypothesis: \(p(n)\) is true and there exists two positive integers \(a_n\) and \(b_n\) such that \(n = 5a_n + 6b_n\)

Then:
\[n + 1 = 5a_n + 6b_n + 1\]

Since 1 can be written as \(6 - 5\) we can write
\[n + 1 = 5a_n + 6b_n + 6 - 5 = 5(a_n - 1) + 6(b_n + 1)\]

Since \(b_n\) is greater than or equal to 0, then \((b_n + 1)\) is also greater than 0

\((a_n - 1)\) is only positive if \(a_n\) is greater or equal to 1.

There are therefore two situations that we need to consider: \(a_n \geq 1\) and \(a_n = 0\).

i) \(a_n \geq 1\)

Then \(n + 1\) can be written as:
\[n + 1 = 5(a_n - 1) + 6(b_n + 1)\]

where both \((a_n - 1)\) and \((b_n + 1)\) are positive.

We can set \(a_{n+1} = a_n - 1\) and \(b_{n+1} = b_n + 1\). In this case, \(p(n + 1)\) is true.

ii) \(a_n = 0\)

\[n + 1 = 6b_n + 1\]
\[n + 1 = 6b_n + 25 - 24\]
\[n + 1 = 5 \times 5 + 6(b_n - 4)\]

\(n + 1\) can be written as 5 times a positive integer 5 and 6 times \((b_n - 4)\).

Notice that \(n = 6b_n\). Since \(n > 20\), \(6b_n > 20\). Since \(b_n\) is an integer, we conclude that \(b_n \geq 4\). Therefore \((b_n - 4) \geq 0\).

We can set \(a_{n+1} = 5\) and \(b_{n+1} = b_n - 4\). In this case, \(p(n + 1)\) is true
In all cases, we have proven that \( p(n + 1) \) is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that \( p(n) \) is true for all \( n > 20 \).

3) Show that:

\[
\forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} (-1)^{i}i^2 = \frac{(-1)^n n(n+1)}{2}
\]

We want to prove: \( P(n) \) is true, for all \( n \geq 1 \), where

\[
P(n): \sum_{i=1}^{n} (-1)^{i}i^2 = \frac{(-1)^n n(n+1)}{2}
\]

Let us define \( LHS(n) = \sum_{i=1}^{n} (-1)^{i}i^2 \), and \( RHS(n) = \frac{(-1)^n n(n+1)}{2} \).

- **Basis step**: We want to prove \( P(1) \) is true.
  \[
  LHS(1) = (-1)^1 1^2 = -1,
  \]
  and
  \[
  RHS(1) = \frac{(-1)^1 \times 1 \times 2}{2} = -1.
  \]
  Therefore, \( LHS(1) = RHS(1) \): \( P(1) \) is true.

- **Inductive step**: Let \( P(n) \) be true for an integer \( n \geq 1 \), which means \( LHS(n) = RHS(n) \).
  To prove that \( P(n+1) \) is true, we prove that \( LHS(n+1) = RHS(n+1) \). Let us compute \( LHS(n+1) \):

  \[
  LHS(n + 1) = \sum_{i=1}^{n+1} (-1)^{i}i^2
  = \sum_{i=1}^{n} (-1)^{i}i^2 + (-1)^{n+1}(n + 1)^2
  = LHS(n) + (-1)^{n+1}(n + 1)^2
  = RHS(n) + (-1)^{n+1}(n + 1)^2
  = \frac{(-1)^n n(n+1)}{2} + (-1)^{n+1}(n + 1)^2
  = \frac{(-1)^n n(n + 1) + 2(-1)^{n+1}(n + 1)^2}{2}
  = \frac{(-1)^{n+1}(n + 1)(-n + 2(n + 1))}{2}
  = \frac{(-1)^{n+1}(n + 1)(n + 2)}{2}
  \]

  and

  \[
  RHS(n + 1) = \frac{(-1)^{n+1}(n + 1)(n + 2)}{2}
  \]

Therefore \( LHS(n + 1) = RHS(n + 1) \), i.e. \( P(n + 1) \) is true.
According to the principle of mathematical induction, we can conclude that 
\[ \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2} \] 
for all \( n \geq 1 \).

4) Let \( f_n \) be the \( n \)-th Fibonacci number (note: Fibonacci numbers satisfy \( f_0 = 0, f_1 = 1 \) and \( f_n + f_{n+1} = f_{n+2} \)). Prove by induction that for all natural numbers \( n \geq 3 \),
\[ \frac{f_1}{f_2f_3} + \frac{f_2}{f_3f_4} + \ldots + \frac{f_{n-2}}{f_{n-1}f_n} = 1 - \frac{1}{f_n} \]

We want to prove: \( P(n) \) is true, for all \( n \geq 3 \), where
\( P(n) : \frac{f_1}{f_2f_3} + \frac{f_2}{f_3f_4} + \ldots + \frac{f_{n-2}}{f_{n-1}f_n} = 1 - \frac{1}{f_n} \)

Let us define \( LHS(n) = \frac{f_1}{f_2f_3} + \frac{f_2}{f_3f_4} + \ldots + \frac{f_{n-2}}{f_{n-1}f_n} \), and \( RHS(n) = 1 - \frac{1}{f_n} \).

- **Basis step:** We want to prove \( P(3) \) is true.
  \[ LHS(3) = \frac{f_1}{f_2f_3} = \frac{1}{1 \times 2} = \frac{1}{2} \]
  and
  \[ RHS(3) = 1 - \frac{1}{f_3} = 1 - \frac{1}{2} = \frac{1}{2} \]

Therefore, \( LHS(3) = RHS(3) \): \( P(3) \) is true.

- **Inductive step:** Let \( P(n) \) be true for an integer \( n \geq 3 \), which means \( LHS(n) = RHS(n) \). To prove that \( P(n+1) \) is true, we prove that \( LHS(n+1) = RHS(n+1) \). Let us compute \( LHS(n+1) \):

\[
LHS(n+1) = \frac{f_1}{f_2f_3} + \frac{f_2}{f_3f_4} + \ldots + \frac{f_{n-2}}{f_{n-1}f_n} + \frac{f_{n-1}}{f_{n}f_{n+1}}
\]

\[
= LHS(n) + \frac{f_{n-1}}{f_{n}f_{n+1}}
\]

\[
= RHS(n) + \frac{f_{n-1}}{f_{n}f_{n+1}}
\]

\[
= 1 - \frac{1}{f_n} + \frac{f_{n-1}}{f_{n}f_{n+1}}
\]

\[
= 1 - \frac{f_{n+1} - f_{n-1}}{f_{n}f_{n+1}}
\]

\[
= 1 - \frac{f_{n+1}}{f_{n}f_{n+1}} + \frac{f_{n-1}}{f_{n}f_{n+1}}
\]

\[
= 1 - \frac{1}{f_n}
\]

and

\[
RHS(n+1) = 1 - \frac{1}{f_{n+1}}
\]
Therefore $\text{LHS}(n + 1) = \text{RHS}(n + 1)$, i.e. $P(n + 1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 3$. 