Midterm 2: solutions

Exercise 1 (2 questions, 20 points total)

Let $n$ be an integer. Give a direct proof and an indirect proof of the proposition, if $n$ is odd then $2n^2 + 5n + 2$ is odd

We want to prove an implication of the form $p \rightarrow q$ is true, with:

$p$: $n$ is odd

$\neg p$: $n$ is even

$q$: $2n^2 + 5n + 2$ is odd

$\neg q$: $2n^2 + 5n + 2$ is even

We use two methods of proof:

a) Direct proof: we show $p \rightarrow q$ is true.

Let us assume that $p$ is true, i.e. that $n$ is odd. There exists an integer $k$ such that $n = 2k + 1$. Therefore,

$$2n^2 + 5n + 2 = 2(2k + 1)^2 + 5(2k + 1) + 2 = 8k^2 + 18k + 9 = 2(4k^2 + 9k + 4) + 1$$

As $k$ is an integer, $4k^2 + 9k + 4$ is an integer which we call $l$. Therefore $2n^2 + 5n + 2 = 2l + 1$, i.e. it is odd.

We have shown that $q$ is true when $p$ is true: the proposition $p \rightarrow q$ is true.

b) Indirect proof: we show $\neg q \rightarrow \neg p$ is true.

Let us assume that $\neg q$ is true, i.e. that $2n^2 + 5n + 2$ is even. There exists an integer $k$ such that $2n^2 + 5n + 2 = 2k$. Therefore,

$$2n^2 + 4n + n + 2 = 2k$$

$$n = 2k - 2n^2 - 4n - 2 = 2(k - n^2 - 2n - 1)$$
As \( k \) and \( n \) are integers, \( k - n^2 - 2n - 1 \) is an integer which we call \( l \). Therefore \( n = 2l \), i.e. it is even.

We have shown that \( \neg p \) is true when \( \neg q \) is true: the proposition \( \neg q \rightarrow \neg p \) is true and, by equivalence, \( p \rightarrow q \) is true.

**Exercise 2 (1 question, 10 points)**

Let \( m \) and \( n \) be 2 integers. Using the method of proof of your choice, show that if \( mn \) is odd, then \( m \) is odd and \( n \) is odd.

We want to prove an implication of the form \( p \rightarrow q \) is true, with:

\( p: \) \( mn \) is odd
\( \neg p: \) \( mn \) is even
\( q: \) \( m \) is odd and \( n \) is odd
\( \neg q: \) \( m \) is even or \( n \) is even

We use an indirect proof: we show that \( \neg q \rightarrow \neg p \) is true.

Let us assume that \( \neg q \) is true, namely that \( m \) is even or \( n \) is even. We consider two cases:

a)\( m \) is even. There exists an integer \( k \) such that \( m = 2k \). Then,

\[
mn = 2kn = 2(kn)
\]

As \( k \) and \( n \) are integers, \( kn \) is an integer which we call \( l \). Therefore \( mn = 2l \), i.e. it is even.

b)\( n \) is even. There exists an integer \( k \) such that \( n = 2k \). Then,

\[
mn = 2km = 2(km)
\]

As \( k \) and \( m \) are integers, \( km \) is an integer which we call \( l \). Therefore \( mn = 2l \), i.e. it is even.

In both cases, we have shown that \( mn \) is even. Therefore \( \neg p \) is true when \( \neg q \) is true. the proposition \( \neg q \rightarrow \neg p \) is true and, by equivalence, \( p \rightarrow q \) is true.

**Exercise 3 (1 question, 10 points)**

Let \( n \) be an integer. Use a proof by contradiction to show that \( \frac{6n+1}{2n+4} \) is not an integer.

Let:

\( P: \) \( \frac{6n+1}{2n+4} \) is not an integer

We use a proof by contradiction. We assume that \( P \) is false, i.e. we assume that \( \frac{6n+1}{2n+4} \) is an integer. Let us name this integer as \( k \). We have:

\[
\frac{6n+1}{2n+4} = k
\]
which we rewrite as:

\[ 6n + 1 = k(2n + 4) \]

Let \( LHS = 6n + 1 \) and \( RHS = k(2n + 4) \). Notice that:

\[ LHS = 2(3n) + 1 \]

Since \( n \) is an integer, \( 3n \) is an integer and therefore \( LHS \) is odd. Conversely,

\[ RHS = 2(k(n + 2)) \]

As \( k \) and \( n \) are integers, \( k(n + 2) \) is an integer which we call \( l \). Therefore \( RHS = 2l \), i.e. it is even.

Under the assumption that \( P \) is false, we find that \( LHS = RHS \) with \( LHS \) odd and \( RHS \) even. Since an even number cannot be equal to an odd number, we have reached a contradiction. Therefore the assumption that \( P \) is false, is false, i.e. \( P \) is true.

**Exercise 4 (1 question, 10 points)**

Let \( n \) be a natural number (i.e., \( n \) is a positive integer different from 0). Use a proof by contradiction to show that if \( n \) is a perfect square, then \( 2n \) is not a perfect square. (A natural number \( n \) is a perfect square if and only if there exists an integer \( k \) such that \( n = k^2 \)).

We want to prove an implication of the form \( p \rightarrow q \) is true, with:

\[ p: n \text{ is a perfect square} \]
\[ \neg p: n \text{ is not a perfect square} \]
\[ q: 2n \text{ is not a perfect square} \]
\[ \neg q: 2n \text{ is a perfect square} \]

We use a proof by contradiction. We assume that \( p \rightarrow q \) is false, i.e. that \( p \) is true AND \( q \) is false.

Since \( p \) is true, \( n \) is a perfect square: there exists an integer \( k \) such that \( n = k^2 \).

Since \( q \) is false, \( 2n \) is a perfect square: there exists an integer \( l \) such that \( 2n = l^2 \).

Replacing \( n \) by \( k^2 \), we get:

\[ 2k^2 = l^2 \]

As \( n \) is non zero, \( l \) is not zero. Therefore:

\[ 2 = \frac{k^2}{l^2} \]

Taking the square root (the numbers are now real),

\[ \sqrt{2} = \frac{|k|}{|l|} \]

As \( k \) is an integer, \( |k| \) is an integer. Similarly, as \( l \) is an integer, \( |l| \) is an integer. This would lead to \( \sqrt{2} \) is rational: this is a contradiction, as we know that \( \sqrt{2} \) is irrational.

Therefore the assumption that \( p \rightarrow q \) is false, is false, i.e. \( p \rightarrow q \) is true.
Exercise 5 (1 question, 10 points)

Let $x$ be a real number. Show that if $x^3 + x^2 - 2x < 0$, then $x < 1$.

We want to prove an implication of the form $p \rightarrow q$ is true, with:

$p$: $x^3 + x^2 - 2x < 0$

$\neg p$: $x^3 + x^2 - 2x \geq 0$

$q$: $x < 1$

$\neg q$: $x \geq 1$

We use an indirect proof, i.e. we prove that $\neg q \rightarrow \neg p$ is true. We assume that $\neg q$ is true, i.e. that $x \geq 1$.

Let $A = x^3 + x^2 - 2x$. Notice that,

$$A = x(x-1)(x+2)$$

We know that:

i) $x > 0$ since $x \geq 1$

ii) $x-1 \geq 0$ since $x \geq 1$

iii) $x+2 > 0$ since $x \geq 1$

The three terms in $A$ are positive: $A$ is positive. Therefore $\neg p$ is true.

We have shown that $\neg p$ is true when $\neg q$ is true. the proposition $\neg q \rightarrow \neg p$ is true and, by equivalence, $p \rightarrow q$ is true.

Exercise 6 (1 question, 10 points)

Prove or disprove that there exists an integer $n$ such that $n^2 + 3n + 2$ is odd.

Let:

$P :$ There exists an integer $n$ such that $n^2 + 3n + 2$ is odd

$P$ is likely to be false. To prove that it is false, we need to show that $\neg P$ is true, namely that

$\neg P :$ For all integers $n$, $n^2 + 3n + 2$ is even.

We use a proof by case:

case a) $n$ is even.

There exists an integer $k$ such that $n = 2k$. Then,

$$n^2 + 3n + 2 = (2k)^2 + 3(2k) + 2$$

$$= 4k^2 + 6k + 2$$

$$= 2(2k^2 + 3k + 1)$$

As $k$ is an integer, $2k^2 + 3k + 1$ is an integer which we call $l$. Therefore $n^2 + 3n + 2 = 2l$, i.e. it is even.
case b) $n$ is odd.

There exists an integer $k$ such that $n = 2k + 1$. Then,

$$n^2 + 3n + 2 = (2k + 1)^2 + 3(2k + 1) + 2$$
$$= 4k^2 + 4k + 1 + 6k + 3 + 2$$
$$= 2(2k^2 + 5k + 3)$$

As $k$ is an integer, $2k^2 + 5k + 3$ is an integer which we call $l$. Therefore $n^2 + 3n + 2 = 2l$, i.e. it is even.

In all cases, $n^2 + 3n + 2$ is even.

We have shown that $\neg P$ is true, therefore the original proposition $P$ is false.