Homework 1 - For 1/18/2023

Exercise 1 (10 points)

Fifty-six biscuits are to be fed to ten pets; each pet is either a cat or a dog. Each dog is to get six biscuits, and each cat is to get five. How many dogs and how many cats are there?

It is best to formalize the problem. Let $C$ be the number of cats, and $D$ the number of dogs. We translate the information we have as relationships on $C$ and $D$:

- There are 10 pets total: $C + D = 10$
- Each dog eats 6 biscuits, each cat eats 5 biscuits, and there are 56 biscuits: $6D + 5C = 56$

The corresponding system of equation

\[
\begin{align*}
C + D &= 10 \\
6D + 5C &= 56
\end{align*}
\]

leads to $D = 6$ and $C = 4$.

Interestingly, there is another way to solve this problem. Each pet eats at least 5 biscuits. As there are 10 pets, this leads to 50 biscuits, and therefore there are only 6 biscuits left. Those biscuits are then given to dogs, one per dog... and therefore there are 6 dogs, and consequently 4 cats.

Exercise 2 (10 points)

A baseball and a baseball bat cost $1.10. The bat costs $1.0 more than the ball. How much does the ball cost?

Intuitively, we would say the bat costs $1.0 and the ball costs $0.10 but we would be wrong!. We need a more systematic approach to solve this problem. Let $x$ be the cost of the ball, and $y$ the cost of the bat. We translate the two sentences given to us as:

\[
\begin{align*}
x + y &= 1.10 \\
y &= x + 1.0
\end{align*}
\]

The solution to this system of equations is $x = $0.05 and $y = $1.05.
Exercise 3 (10 points)

There are 100 cats living in a barn, all are either brown or orange. There is at least one orange cat. For every two cats, at least one is brown. How many orange cats and how many brown cats are there?

We know that there is at least one orange cat. Let us call it $OR$. Consider any other of the 99 cats, $C$. The pair $(OR, C)$ is a pair of cats. Based on the problem, this pair contains at least one brown cat. Since $OR$ is orange, $C$ is then brown, and this is true for all the 99 cats besides $OR$.

Therefore, there is one orange cat and there are 99 brown cats.

Exercise 4 (5 points each; total 20 points)

Prove the following identities for $p, q, m, n, x,$ and $y$ real numbers:

a) $8(p - q) + 4(p + q) = 2(p + 3q) + 10(p - q)$

Let $p$ and $q$ be two real numbers, and let $LHS = 8(p - q) + 4(p + q)$ and $RHS = 2(p + 3q) + 10(p - q)$. Then:

$LHS = 8p - 8q + 4p + 4q$

$= 12p - 4q$

and

$RHS = 2p + 6q + 10p - 10q$

$= 12p - 4q$

Therefore $LHS = RHS$ for all $p$ and $q$, and the identity is true.

b) $x(m - n) + y(n + m) = m(x + y) + n(y - x)$

Let $x, y, m$ and $n$ be four real numbers, and let $LHS = x(m - n) + y(n + m)$ and $RHS = m(x + y) + n(y - x)$. Then:

$LHS = xm - xn + yn + ym$

and

$RHS = xm - xn + yn + ym$

Therefore $LHS = RHS$ for all $x, y, n$ and $m$, and the identity is true.

c) $(x + 3)(x + 8) - (x - 6)(x - 4) = 21x$

Let $x$ be a real number and let $LHS = (x + 3)(x + 8) - (x - 6)(x - 4)$ and $RHS = 21x$. Then:

$LHS = x^2 + 8x + 3x + 24 - x^2 + 4x + 6x - 24$

$= 21x$

$= RHS$

The identity is true for all $x$. 


d) \[ m^8 - 1 = (m^2 - 1)(m^2 + 1)(m^4 + 1) \]

Let \( m \) be a real number and let \( LHS = m^8 - 1 \) and \( RHS = (m^2 - 1)(m^2 + 1)(m^4 + 1) \). Then

\[
LHS = (m^4)^2 - 1^2 = (m^4 - 1)(m^4 + 1) = ((m^2)^2 - 1)(m^4 + 1) = (m^2 - 1)(m^2 + 1)(m^4 + 1) = RHS
\]

The identity is true for all \( m \).

**Exercise 5 (10 points)**

Let us play a logical game. You find yourself in front of three rooms whose doors are closed. One of these rooms contains a Lady, another a Tiger and the third room is empty. There is one sign on each door; you are told that the sign on the door of the room containing the Lady is true, the sign on the door of the room with the Tiger is false, and the sign on the door of the empty room could be either true or false. Here are the signs:

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Room III is empty</td>
<td>The Tiger is in room I</td>
<td>This room is empty</td>
</tr>
</tbody>
</table>

Figure 1: The three rooms and their signs

*Can you find what is behind each door?*

We solve this problem using a table, just like for the knights and knaves problem. We do not know in which order these rooms are, but we do know we have all three (i.e. we do not have 2 rooms containing a Tiger for example): there are 6 different ways to organize these three rooms. For each way, we analyze the three signs given and indicate if they are true (T) or false (F):

<table>
<thead>
<tr>
<th>Line</th>
<th>Room I</th>
<th>Room II</th>
<th>Room III</th>
<th>Sign I</th>
<th>Sign II</th>
<th>sign III</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Lady</td>
<td>Tiger</td>
<td>Empty</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>2</td>
<td>Lady</td>
<td>Empty</td>
<td>Tiger</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>3</td>
<td>Tiger</td>
<td>Lady</td>
<td>Empty</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>4</td>
<td>Tiger</td>
<td>Empty</td>
<td>Lady</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>5</td>
<td>Empty</td>
<td>Lady</td>
<td>Tiger</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>6</td>
<td>Empty</td>
<td>Tiger</td>
<td>Lady</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Now we note that:

- line 1 is consistent.
• line 2 cannot be correct: one of the signs must be true as one room contains the Lady
• line 3 cannot be correct: one of the signs must be false as one room contains the Tiger
• line 4 cannot be correct: the sign on the Lady’s room would be false.
• line 5 cannot be correct: one of the signs must be true as one room contains the Lady
• line 6 cannot be correct: one of the signs must be true as one room contains the Lady

Line 1 is the only valid possibility, therefore the Lady is in room I, the tiger in room II, and room III is empty.

Exercise 6 (10 points)

You are on an island inhabited by three types of people: knights (always make true statements), knaves (always make false statements) and spies (sometimes make true statements and sometimes make false statements). You come across 3 people Adam, Ben and Carl. You know that one is a knight, one is a knave, and one is a spy. They say the following:

• Adam: “I am not a knight.”
• Ben: “I am not a knave.”
• Carl: “I am not a spy.”

Determine which person is what or whether you do not have enough information.

Again, the easiest is to use a table: T means that the person said the truth, and F that the person said a lie. Note that when we fill up the table, we need to consider that we have one Knight, one Knave, and one Spy.

<table>
<thead>
<tr>
<th>Line</th>
<th>Adam</th>
<th>Ben</th>
<th>Carl</th>
<th>Adam says</th>
<th>Ben says</th>
<th>Carl says</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Knight</td>
<td>Knave</td>
<td>Spy</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>Knight</td>
<td>Spy</td>
<td>Knave</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>3</td>
<td>Knave</td>
<td>Knight</td>
<td>Spy</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>Knave</td>
<td>Spy</td>
<td>Knight</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>5</td>
<td>Spy</td>
<td>Knight</td>
<td>Knave</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>6</td>
<td>Spy</td>
<td>Knave</td>
<td>Knight</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Now we note that:
• lines 1 and 2 cannot be correct: Adam would be a knight that lies.
• line 3 and 4 cannot be correct: Adam would be a knave that tells the truth.
• line 5 cannot be correct: Carl would be a knave that tells the truth

Line 6 is the only possibility, i.e. Adam is a Spy, Ben is a Knave, and Carl is a Knight.