Exercise 1 (10 points)

Let us play a logical game. You find yourself in front of three rooms whose doors are closed. One of these rooms contains a Lady, another a Tiger and the third room is empty. There is one sign on each door; you are told that the sign on the door of the room containing the Lady is true, the sign on the door of the room with the Tiger is false, and the sign on the door of the empty room could be either true or false. Here are the signs:

I
Room III is empty

II
The Tiger is in room I

III
This room is empty

Figure 1: The three rooms and their signs

We solve this problem using a table, just like for the knights and knaves problem. Let us define the symbol LA, TI and EM for the room containing the Lady, the Tiger and being Empty, respectively. We do not know in which order these rooms are, but we do know we have all three (i.e. we do not have 2 rooms containing a Tiger for example): there are 6 different ways to organize these three rooms. For each way, we analyze the three signs given and indicate if they are true (T) or false (F):

<table>
<thead>
<tr>
<th>Line</th>
<th>Room I</th>
<th>Room II</th>
<th>Room III</th>
<th>Sign I</th>
<th>Sign II</th>
<th>sign III</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LA</td>
<td>TI</td>
<td>EM</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>2</td>
<td>LA</td>
<td>EM</td>
<td>TI</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>3</td>
<td>TI</td>
<td>LA</td>
<td>EM</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>4</td>
<td>TI</td>
<td>EM</td>
<td>LA</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>5</td>
<td>EM</td>
<td>LA</td>
<td>TI</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>6</td>
<td>EM</td>
<td>TI</td>
<td>LA</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Now we note that:

- line 1 is consistent.
- line 2 cannot be correct: one of the signs must be true as one room contains the Lady
- line 3 cannot be correct: one of the signs must be false as one room contains the Tiger
- line 4 cannot be correct: the sign on the Lady’s room would be false.
- line 5 cannot be correct: one of the signs must be true as one room contains the Lady
- line 6 cannot be correct: one of the signs must be true as one room contains the Lady

Line 1 is the only valid possibility, therefore the Lady is in room I, the tiger in room II, and room III is empty.

**Exercise 2 (10 points)**

*We are on the island of knights and knaves. You meet two residents, John and Bill who make the following statements:*

John says: “If Bill is a knave, then I am a knight”

Bill says: “We are different”

*Can you find what are John and Bill?*

We check all possible “values” for John and Bill, as well as the veracity of their statements. Note that John’s statement is an implication.

<table>
<thead>
<tr>
<th>Line number</th>
<th>John</th>
<th>Bill</th>
<th>John says</th>
<th>Bill says</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Knight</td>
<td>Knight</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>Knight</td>
<td>Knave</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>3</td>
<td>Knave</td>
<td>Knight</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>4</td>
<td>Knave</td>
<td>Knave</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

We can eliminate:

- Line 1, as Bill would be a knight but he lies
- Line 2, as Bill would be a knave but he tells the truth
- Line 3 as John would be a knave but he says the true

Line 4 is valid, and it is the only one. Therefore, both John and Bill are knaves.
**Exercise 3** (10 points)

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, “At least one of the following is true: that Sally is a knave or that I am a knight.” John says, ”Alex could claim that I am a knave.” Sally claims, “Neither Alex nor John are knights.” Can you find who is a knight and who is a knave?

We check all possible "values" for Alex, John and Sally, as well as the veracity of their statements. Note that Alex’s statement is based on OR, John’s statement is dependent on Alex’s type and Sally statement’s is based on NAND.

<table>
<thead>
<tr>
<th>Line number</th>
<th>Alex</th>
<th>John</th>
<th>Sally</th>
<th>Alex says</th>
<th>John says</th>
<th>Sally says</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Knight</td>
<td>Knight</td>
<td>Knight</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>Knight</td>
<td>Knight</td>
<td>Knave</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>3</td>
<td>Knight</td>
<td>Knave</td>
<td>Knight</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>Knight</td>
<td>Knave</td>
<td>Knave</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>5</td>
<td>Knave</td>
<td>Knight</td>
<td>Knight</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>6</td>
<td>Knave</td>
<td>Knight</td>
<td>Knave</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>7</td>
<td>Knave</td>
<td>Knave</td>
<td>Knight</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>8</td>
<td>Knave</td>
<td>Knave</td>
<td>Knave</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

We can eliminate:

- Line 1, as John would be a knight but he lies
- Line 2, as John would be a knight but he lies
- Line 3 as John would be a knave but he says the true
- Line 4 as John would be a knave but he says the true
- Line 5 as Sally would be a knight but she lies
- Line 6 as Alex would be a knave but he says the true
- Line 8 as Alex would be a knave but he says the true

Line 7 is valid, and it is the only one. Therefore, both Alex and John are knaves, while Sally is a knight.

**Exercise 4** (10 points)

Design a circuit that implements majority voting for three individuals (i.e. the output of the circuit is 1 if two at least of the inputs are 1, and 0 otherwise). (Hint: consider the Boolean expression $(A + B) \cdot C + (A + C) \cdot B$).

One solution is: To check that this is what we need, we build its truth table:

$0 = (A + B) \cdot C + (A + C) \cdot B$ is the expected output.
Exercise 5 (2 parts, each 10 points; Total 20 points)

a) Assume that you only have NAND gate at your disposal. Show that the following gate is equivalent to the OR gate.

We build the truth table for this gate:
The output of this gate is fully equivalent to the OR gate.

b) Draw a diagram whereby multiple NAND gates are connected together to form an AND gate. (Hint: the NOT gate can be formed using:)

We build its truth table to be sure:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$O$</th>
<th>$A \cdot B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The output of this gate is fully equivalent to the AND gate.

**Exercise 6 (10 points)**

*Show that the proposition $P = [(r \lor p) \rightarrow (r \lor q)] \leftrightarrow [r \lor (p \rightarrow q)]$ is a tautology.*

We use a truth table:
Therefore $P$ is a tautology.

**Exercise 7 (10 points)**

Show that the following is a tautology: \[(p \lor q) \land (p \to r) \land (q \to r)\] $\to r$.

Let us define $A = (p \lor q) \land (p \to r) \land (q \to r)$. The proposition is $A \to r$.

\[
\begin{array}{cccccccccccc}
  p & q & r & p \lor p & r \lor q & (r \lor p) \to (r \lor q) & p \to q & r \lor (p \to q) & P \\
  T & T & T & T & T & T & T & T & T \\
  T & T & F & T & T & T & T & T & T \\
  T & F & T & T & T & F & T & T & T \\
  T & F & F & T & F & F & F & F & T \\
  F & T & T & T & T & T & T & T & T \\
  F & T & F & T & T & T & T & T & T \\
  F & F & T & T & T & T & T & T & T \\
  F & F & F & T & T & T & T & T & T \\
\end{array}
\]

Therefore $P = A \to r$ is a tautology.