Exercise 1 (10 points)

Prove that if \( n \) is a positive integer, then \( n \) is even if and only if \( 7n + 4 \) is even.

Let \( p \) be the proposition “\( n \) is even” and \( q \) be the proposition “\( 7n + 4 \) is even”. We want to show that \( p \leftrightarrow q \) is true, which is logically equivalent to show that \( p \rightarrow q \) and \( q \rightarrow p \).

i) Let us show \( p \rightarrow q \):

Hypothesis: \( p \) is true, i.e. \( n \) is even. If \( n \) is even, then there exists an integer \( k \) such that let \( n = 2k \), We get:

\[
7n + 4 = 7(2k) + 4 = 14k + 4 = 2(7k + 2)
\]

As \( 7k + 2 \) is an integer, \( 7n + 4 \) is a multiple of 2: it is even.

ii) Let us show \( q \rightarrow p \):

Hypothesis: \( q \) is true, i.e. \( 7n + 4 \) is even. If \( 7n + 4 \) is even, then there exists an integer \( k \) such \( 7n + 4 = 2k \). We get:

\[
7n = 2k - 4
\]

\[
n = 2(k - 2 - 3n)
\]

As \( k - 2 - 3n \) is an integer, \( n \) is a multiple of 2: it is even. Note that we could have done this proof using a contraposition.

We conclude: “\( n \) is even” and “\( 7n + 4 \) is even” are logically equivalent.
**Exercise 2 (10 points)**

Let $a$ and $b$ be two integers. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Let $p$ be the proposition “$n = ab$”, and let $q$ be the proposition $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. We use a proof by contradiction, namely we assume that $p$ is true AND $q$ is false.

Since $q$ is false, we know that

\begin{align*}
a & > \sqrt{n} \quad (1) \\
b & > \sqrt{n} \quad (2)
\end{align*}

As $\sqrt{n}$ is positive, we have that both $a > 0$ and $b > 0$. We multiply (1) with $b$ and (2) with $\sqrt{n}$:

\begin{align*}
ab & > b\sqrt{n} \quad (3) \\
b\sqrt{n} & > n \quad (4)
\end{align*}

By transitivity, we get that $ab > \sqrt{n}$, but we also have $ab = n$ as $p$ is true. We have reached a contradiction. Therefore the original implication is true.

**Exercise 3 (10 points)**

Let $m$ and $n$ be two integers. Show that if $m > 0$ and $n \leq -2$, then $m^2 + mn + n^2 \geq 0$.

We use a direct proof. Let $p$ be the proposition $m > 0$ and $n \leq -2$, and let $q$ be the proposition $m^2 + mn + n^2 \geq 0$. To show $p \rightarrow q$, we will show that if $p$ is true, then $q$ is also true.

Let $m$ and $n$ be two integers.

Hypothesis: $p$ is true. Therefore $m > 0$ and $n \leq -2$. As $m$ is (strictly) positive, we can multiply $n \leq -2$ by $m$ without changing the sense of the inequality; then $mn \leq -2m$. As $m$ is strictly positive, $-2m$ is strictly negative. Therefore:

\begin{align*}
mn & \leq -2m \\
-2m & < 0
\end{align*}

i.e. $mn < 0$.

Let us consider now:

\begin{align*}
m^2 + mn + n^2 &= m^2 + 2mn + n^2 - mn \\
&= (m + n)^2 - mn
\end{align*}

Note that $(m + n)^2$ is positive and $-mn$ is also positive, as $mn$ is negative. $m^2 + mn + n^2$ is the sum of two positive numbers; it is positive. Therefore $q$ is true, and the property is true.

**Exercise 4 (10 points each; total 30 points)**

Let $a$ and $b$ be two integers. Show that if $a^2 + b^2$ is even, then $a + b$ is even:

We define:

$p$: $a^2 + b^2$ is even

$q$: $a + b$ is even
a) *Using an indirect proof (proof by contrapositive)*

Hypothesis: \( a + b \) is odd. Therefore, there exists an integer \( k \) such that \( a + b = 2k + 1 \). We compute \((a + b)^2\) in two different ways,

\[
\begin{align*}
(a + b)^2 &= a^2 + b^2 + 2ab \\
(a + b)^2 &= 4k^2 + 4k + 1
\end{align*}
\]

Therefore,

\[
\begin{align*}
a^2 + b^2 &= 4k^2 + 4k + 1 - 2ab \\
&= 2(2k^2 + 2k - ab) + 1
\end{align*}
\]

As \(2k^2 + 2k - ab\) is an integer, we get that \(a^2 + b^2\) is odd, i.e. \(\neg p\) is true. This concludes the indirect proof.

b) *Using a proof by contradiction*

Hypothesis: \( p \) is true AND \(\neg q\) is true. Therefore, we know that \(a^2 + b^2\) is even, and \(a + b\) is odd. As \(a + b\) is odd, there exists an integer \( k \) such that \(a + b = 2k + 1\). We compute \((a + b)^2\) in two different ways,

\[
\begin{align*}
(a + b)^2 &= a^2 + b^2 + 2ab \\
(a + b)^2 &= 4k^2 + 4k + 1
\end{align*}
\]

Therefore,

\[
\begin{align*}
a^2 + b^2 &= 4k^2 + 4k + 1 - 2ab \\
&= 2(2k^2 + 2k - ab) + 1
\end{align*}
\]

As \(2k^2 + 2k - ab\) is an integer, we get that \(a^2 + b^2\) is odd, i.e. \(\neg p\) is true. However, we had assumed that \(p\) is true: we have reached a contradiction. This concludes the proof by contradiction.

c) *Using a direct proof*

Hypothesis: \( p \) is true. Then \(a^2 + b^2\) is even, i.e. there exists an integer \( k \) such that \(a^2 + b^2 = 2k\). Note that

\[
\begin{align*}
(a + b)^2 &= a^2 + b^2 + 2ab \\
&= 2k + 2ab \\
&= 2(k + ab)
\end{align*}
\]

As \(k + ab\) is an integer, \((a + b)^2\) is even, and therefore \(a + b\) is even. This concludes the direct proof.

**Exercise 5 (10 points)**

*Show that \(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, 2y = x + x^2\)*

Let \(x\) be an integer. \(x\) is either odd, or even. We check both cases:
Case 1: $x$ is odd.

We show that $x + x^2$ is even using a direct proof. $x + x^2 = 2k + 1 + 2(2k^2 + 2k) + 1 = 2(2k^2 + 3k + 1)$ therefore $x + x^2$ is even.

As $x + x^2$ is even, there exists $l \in \mathbb{Z}$ such that $x + x^2 = 2l$. We set $y = l$: we have shown the existence of $y$ when $x$ is odd.

Case 2: $x$ is even.

We show that $x + x^2$ is even. As $x$ is even, there exists $k \in \mathbb{Z}$ such that $x = 2k$. Then $x + x^2 = 2k + 4k^2 = 2(k + 2k^2)$. Therefore $x + x^2$ is even.

As $x + x^2$ is even, there exists $l \in \mathbb{Z}$ such that $x + x^2 = 2l$. We set $y = l$: we have shown the existence of $y$ when $x$ is even.

We can conclude that there always exists an integer $y$ such that $x + x^2 = 2y$, where $x$ is an integer.

**Exercise 6 (10 points)**

Let $a$ and $b$ be two strictly positive real numbers. Show that $\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$

The only difficulty in this problem was to make sure that we use “good practice”!

Let $a$ and $b$ be two strictly positive real numbers. As both are positive, we can take their square roots. As a square is always positive or zero, $(\sqrt{a} - \sqrt{b})^2 \geq 0$. After developing the square, we get $a + b - 2\sqrt{ab} \geq 0$. As $ab > 0$ (both $a$ and $b$ are positive) we can divide the inequality by $ab$, without changing its sense: $\frac{1}{b} + \frac{1}{a} - \frac{2}{\sqrt{ab}} \geq 0$, i.e. $\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$ which concludes the proof.

**Exercise 7 (10 points)**

You arrive in a country called Transylvania whose inhabitants are humans and vampires. Humans always tell the truth, while vampires always lie. However, both humans and vampires can be sane or insane. If an inhabitant is insane, she will believe that a truth statement is false, and a false statement is true. Sane inhabitants believe that truth statements are true and false statements are false. Thus sane humans and insane vampires make only true statements, while insane humans and sane vampires make only false statements. You meet two inhabitants, A and B. You know that one of them is a human, and the other one is a vampire. A tells you: “we are both insane”, while B tells you that “at least one of us is sane”. From this, can you find which one is the vampire?

Just like with all Smullyan’s problems, we build a truth table. A and B can each be a human or a vampire, sane or insane. This would give 16 possibilities for the pair. However, we know that one is human, and the other one is a vampire: this reduces the table to 8 possibilities.

Let $P_A$ be: “We are both insane”.

Let $P_B$ be: “At least one of us is sane”
We can eliminate:

Line 1: A would be a sane human that lies

Line 2: A would be a sane human that lies

Line 3: B would be a sane vampire that tells the truth

Line 4: A would be an insane human that tells the truth

Line 6: B would be an insane human that tells the truth

Line 7: A would be an insane vampire that lies

Both lines 5 and 8 are compatible with the premises. In both cases, B is a human and A is a vampire. We do not know however if they are sane or not.