Homework 5 - For 2/15/2023

Exercise 1: (10 points each; total 20 points)

Let $p$, $q$, and $r$ be three propositions. Using truth tables or logical equivalences, indicate which (if any) of the propositions below are tautologies, contradictions, or neither.

a) $A = (p \land q) \lor r \lor (\neg q \land \neg r) \lor (\neg p \land \neg r)$

Let us do it first using a truth table

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$p \land q$</th>
<th>$\neg q \land \neg r$</th>
<th>$\neg p \land \neg r$</th>
<th>$A$</th>
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The proposition $A$ is a tautology.
Let us do it now using logical equivalences:

\[
A \iff (p \land q) \lor r \lor (\neg q \land \neg r) \lor (\neg p \land \neg r)
\]

\[
\iff (p \land q) \lor r \lor [(\neg q \lor \neg p) \land \neg r]
\]

\[
\iff (p \land q) \lor r \lor [(\neg (p \land q)) \land \neg r] \lor (r \lor \neg r)
\]

\[
\iff (p \land q) \lor (r \lor (\neg (p \land q)) \land T)
\]

\[
\iff (p \land q) \lor (r \lor (\neg (p \land q))
\]

\[
\iff T \lor r
\]

\[
\iff T
\]

The proposition \( A \) is a tautology.

b) \([p \lor (q \rightarrow r)] \rightarrow (p \lor q \lor r)\)

Let us define \( A = p \lor (q \rightarrow r) \) and \( B = p \lor q \lor r \). Let us build the truth table:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( q \rightarrow r )</th>
<th>( A )</th>
<th>( B )</th>
<th>( A \rightarrow B )</th>
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The proposition \( A \rightarrow B \) is neither a tautology nor a contradiction

**Exercise 2 (10 points)**

Prove that if \( n \) is a positive integer, then \( n \) is even if and only if \( 7n + 4 \) is even.

Let \( p \) be the proposition “\( n \) is even” and \( q \) be the proposition “\( 7n+4 \) is even”. We want to show that \( p \iff q \) is true, which is logically equivalent to show that \( p \rightarrow q \) and \( q \rightarrow p \).

i) Let us show \( p \rightarrow q \):

Hypothesis: \( p \) is true, i.e. \( n \) is even. If \( n \) is even, then there exists an integer \( k \) such that let \( n = 2k \), We get:

\[
7n + 4 = 7(2k) + 4
\]

\[
= 14k + 4
\]

\[
= 2(7k + 2)
\]
As $7k + 2$ is an integer, $7n + 4$ is a multiple of 2: it is even.

ii) Let us show $q \rightarrow p$:

Hypothesis: $q$ is true, i.e. $7n + 4$ is even. If $7n + 4$ is even, then there exists an integer $k$ such $7n + 4 = 2k$. We get:

$$7n = 2k - 4$$
$$n = 2k - 4 - 6n$$
$$n = 2 \times (k - 2 - 3n)$$

As $k - 2 - 3n$ is an integer, $n$ is a multiple of 2: it is even. Note that we could have done this proof using a contrapositive.

We conclude: “$n$ is even” and “$7n + 4$ is even” are logically equivalent.

**Exercise 3 (10 points)**

Let $a$ and $b$ be two positive integers. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Let $p$ be the proposition “$n = ab$”, and let $q$ be the proposition $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. We use a proof by contradiction, namely we assume that $p$ is true AND $q$ is false.

Since $q$ is false, we know that

$$a > \sqrt{n} \quad (1)$$
$$b > \sqrt{n}. \quad (2)$$

As $\sqrt{n}$ is positive, we have that both $a > 0$ and $b > 0$. We multiply (1) with $b$ and (2) with $\sqrt{n}$:

$$ab > b\sqrt{n} \quad (3)$$
$$b\sqrt{n} > n. \quad (4)$$

By transitivity, we get that $ab > \sqrt{n}$, but we also have $ab = n$ as $p$ is true. We have reached a contradiction. Therefore the original implication is true.

**Exercise 4 (10 points)**

Let $m$ and $n$ be two integers. Show that if $m > 0$ and $n \leq -2$, then $m^2 + mn + n^2 \geq 0$

We use a direct proof. Let $p$ be the proposition $m > 0$ and $n \leq -2$, and let $q$ be the proposition $m^2 + mn + n^2 \geq 0$. To show $p \rightarrow q$, we will show that if $p$ is true, then $q$ is also true.

Let $m$ and $n$ be two integers.

Hypothesis: $p$ is true. Therefore $m > 0$ and $n \leq -2$. As $m$ is (strictly) positive, we can multiply $n \leq -2$ by $m$ without changing the sense of the inequality; then $mn \leq -2m$. As $m$ is strictly positive, $-2m$ is strictly negative. Therefore:

$$mn \leq -2m$$
$$-2m < 0$$
i.e. $mn < 0$.

Let us consider now:

$$m^2 + mn + n^2 = m^2 + 2mn + n^2 - mn$$

$$= (m + n)^2 - mn$$

Note that $(m + n)^2$ is positive and $-mn$ is also positive, as $mn$ is negative. $m^2 + mn + n^2$ is the sum of two positive numbers; it is positive. Therefore $q$ is true, and the property is true.

**Exercise 5 (10 points each; total 30 points)**

Let $a$ and $b$ be two integers. Show that if $a^2 + b^2$ is even, then $a + b$ is even:

We define:

$p$: $a^2 + b^2$ is even
$q$: $a + b$ is even

a) *Using an indirect proof (proof by contrapositive)*

Hypothesis: $a + b$ is odd. Therefore, there exists an integer $k$ such that $a + b = 2k + 1$. We compute $(a + b)^2$ in two different ways,

\[
(a + b)^2 = a^2 + b^2 + 2ab \\
(a + b)^2 = 4k^2 + 4k + 1
\]

Therefore,

\[
a^2 + b^2 = 4k^2 + 4k + 1 - 2ab \\
= 2(2k^2 + 2k - ab) + 1
\]

As $2k^2 + 2k - ab$ is an integer, we get that $a^2 + b^2$ is odd, i.e. $\neg p$ is true. This concludes the indirect proof.

b) *Using a proof by contradiction*

Hypothesis: $p$ is true AND $\neg q$ is true. Therefore, we know that $a^2 + b^2$ is even, and $a + b$ is odd. As $a + b$ is odd, there exists an integer $k$ such that $a + b = 2k + 1$. We compute $(a + b)^2$ in two different ways,

\[
(a + b)^2 = a^2 + b^2 + 2ab \\
(a + b)^2 = 4k^2 + 4k + 1
\]

Therefore,

\[
a^2 + b^2 = 4k^2 + 4k + 1 - 2ab \\
= 2(2k^2 + 2k - ab) + 1
\]

As $2k^2 + 2k - ab$ is an integer, we get that $a^2 + b^2$ is odd, i.e. $\neg p$ is true. However, we had assumed that $p$ is true: we have reached a contradiction. This concludes the proof by contradiction.
c) Using a direct proof

Hypothesis: $p$ is true. Then $a^2 + b^2$ is even, i.e. there exists an integer $k$ such that $a^2 + b^2 = 2k$. Note that

\[(a + b)^2 = a^2 + b^2 + 2ab = 2k + 2ab = 2(k + ab)\]

As $k + ab$ is an integer, $(a + b)^2$ is even, and therefore $a + b$ is even. This concludes the direct proof.

Exercise 6 (10 points)

The Fair Maiden Rowena wishes to wed. And her father, the Evil King Berman, has devised a way to drive off suitors. He has a little quiz for them, and here it is. It’s very simple:

Three boxes sit on a table. The first is made of gold, the second is made of silver, and the third is made of lead. Inside one of these boxes is a picture of the fair Rowena. It is the job of the White Knight to figure out, without opening them, which one has her picture.

Now, to assist him in this endeavor there is an inscription on each of the boxes. The gold box says, “Rowena’s picture is in this box.” The silver box says, “The picture is not in this box.” The lead box says, “The picture is not in the gold box.” Only one of the statements is true. Which box holds the picture?

The simplest approach to solve this problem is to check systematically if the Gold box, the Silver box, or the Lead box contains the picture. In each case, we test the validities of the three statements.

<table>
<thead>
<tr>
<th>Box with picture</th>
<th>Golden Box picture is in this box</th>
<th>Silver Box picture is not in this box</th>
<th>Lead Box picture is not in Gold box</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gold</td>
<td>True</td>
<td>True</td>
<td>False</td>
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<tr>
<td>Silver</td>
<td>False</td>
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<td>True</td>
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<tr>
<td>Lead</td>
<td>False</td>
<td>True</td>
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If the prize were in the Gold or Lead box, two of the propositions would be true, whereas if the prize is in Silver Box, only one proposition would be true. The latter is therefore true, and the prize is in the Silver Box.