Homework 6 - For 2/16/2022

Exercise 1 (5 points each; total 20 points)

Determine the truth values of the following statements; justify your answers:

a) \(\forall n \in \mathbb{N}, (n + 2) > n\)

b) \(\exists n \in \mathbb{N}, 2n = 3n\)

c) \(\forall n \in \mathbb{Z}, 3n \leq 4n\)

d) \(\exists x \in \mathbb{R}, x^4 < x^2\)

Exercise 2 (10 points each; total 50 points)

Show that the following statements are true.

a) Let \(x\) be a real number. Prove that if \(x^3\) is irrational, then \(x\) is irrational.

b) Let \(x\) be a positive real number. Prove that if \(x\) is irrational, then \(\sqrt{x}\) is irrational.

c) Prove or disprove that if \(a\) and \(b\) are two rational numbers, then \(a^b\) is also a rational number.

d) let \(n\) be a natural number. Show that \(n\) is even if and only if \(3n + 8\) is even.

e) Prove that either \(4 \times 10^{769} + 22\) or \(4 \times 10^{769} + 23\) is not a perfect square. Is your proof constructive, or non-constructive?

Note: for question e), a natural number \(n\) is a perfect square if there exists a natural number \(q\) such that \(n = q^2\). For example, 4, 9, 16, 25, .... are all perfect squares while 2, 3, 5, 6,... are not.

Exercise 3 (10 points)

Let \(n\) be a natural number and let \(a_1, a_2, \ldots, a_n\) be a set of \(n\) real numbers. Prove that at least one of these numbers is greater than, or equal to the average of these numbers. What kind of proof did you use?
Extra Credit \textit{(10 points)}

Use Exercise 3 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.