Data, Logic, and Computing

ECS 17 (Winter 2023)

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Homework 6 - For 2/22/2023

Exercise 1 (5 points each; total 20)

Determine the truth values of the following statements; justify your answers:

a) \( \forall n \in \mathbb{N}, (n + 2) > n \)
   The statement is True. Let us prove it.
   Let \( n \) be a natural number. Let us define \( A = n + 2 \) and \( B = n \). We notice that \( A - B = n + 2 - n = 2 > 0 \). Therefore, \( A > B \), i.e. \( n + 2 > n \). As this is true for all \( n \), the statement is true.

b) \( \exists n \in \mathbb{N}, 2n = 3n \)
   The statement is False. Let us prove it.
   Let us solve first \( 2n = 3n \) where \( n \) is an integer. We find \( 3n - 2n = 0 \), therefore \( n = 0 \). Therefore, the equation \( 2n = 3n \) is only true for \( n = 0 \). However, 0 does not belong to \( \mathbb{N} \). We can conclude that \( \forall n \in \mathbb{N}, 2n \neq 3n \); the property is false.

c) \( \forall n \in \mathbb{Z}, 3n \leq 4n \)
   The statement is False. Let us prove it.
   Let \( n \) be an integer. \( 3n \leq 4n \) is equivalent to \( 0 \leq n \). This means that \( \forall n < 0, 3n > 4n \). Therefore, we can find \( n \in \mathbb{Z} \) such that \( 3n > 4n \) (for example \( n = -1 \)). The statement is false.

d) \( \exists x \in \mathbb{R}, x^4 < x^2 \)
   The statement is True. Let us prove it.
   Notice that the statement is based on existence: we only need to find one example. if \( x = \frac{1}{2} \). \( x^2 = \frac{1}{4} \) and \( x^4 = \frac{1}{16} \), in which case \( x^4 < x^2 \).

Exercise 2 (10 points each; total 50 points)

Show that the following statements are true.
a) Let $x$ be a real number. Prove that if $x^3$ is irrational, then $x$ is irrational.

Proof: Let $x$ be a real number. We define the two statements: $P(x) : x^3$ is irrational, and $Q(x) : x$ is irrational. We want to show $P(x) \rightarrow Q(x)$. We will prove instead its contrapositive: $\neg Q(x) \rightarrow \neg P(x)$, where $\neg Q(x) : x$ is rational, and $\neg P(x) : x^3$ is rational.

Hypothesis: $\neg Q(x)$ is true, namely $x$ is rational. By definition, there exists two integers $a$ and $b$, with $b \neq 0$, such that $x = \frac{a}{b}$. Then,

$$x^3 = \frac{a^3}{b^3}$$

Since $a$ is an integer, $a^3$ is an integer. Similarly, since $b$ is a non-zero integer, $b^3$ is a non-zero integer. Therefore $x^3$ is rational, which concludes the proof.

b) Let $x$ be a positive real number. Prove that if $x$ is irrational, then $\sqrt{x}$ is irrational.

Proof: Let $x$ be a real number. We define the two statements: $P(x) : x$ is irrational, and $Q(x) : \sqrt{x}$ is irrational. We want to show $P(x) \rightarrow Q(x)$. We will prove instead its contrapositive: $\neg Q(x) \rightarrow \neg P(x)$, where $\neg Q(x) : \sqrt{x}$ is rational, and $\neg P(x) : x$ is rational.

Hypothesis: $\neg Q(x)$ is true, namely $\sqrt{x}$ is rational. By definition, there exists two integers $a$ and $b$, with $b \neq 0$, such that $\sqrt{x} = \frac{a}{b}$. Then,

$$x = \frac{a^2}{b^2}$$

Since $a$ is an integer, $a^2$ is an integer. Similarly, since $b$ is a non-zero integer, $b^2$ is a non-zero integer. Therefore $x$ is rational, which concludes the proof.

c) Prove or disprove that if $a$ and $b$ are two rational numbers, then $a^b$ is also a rational number.

The property is in fact not true. Let $a = 2$ and $b = \frac{1}{2}$. Then $a^b = 2^{\frac{1}{2}} = \sqrt{2}$; but we have shown in class that $\sqrt{2}$ is irrational.

d) Let $n$ be a natural number. Show that $n$ is even if and only if $3n + 8$ is even.

Proof. Let $n$ be a natural number and let $P(n)$ and $Q(n)$ be the propositions $n$ is even, and $3n + 8$ is even, respectively. We will show that $P(n) \rightarrow Q(n)$ and $Q(n) \rightarrow P(n)$.

i) $P(n) \rightarrow Q(n)$

Hypothesis: $n$ is even. By definition of even numbers, there exists and integer $k$ such that $n = 2k$. Then,

$$3n + 8 = 6k + 8 = 2(3k + 4)$$

Since $3k + 4$ is an integer, $3n + 8$ can be written in the form $2k'$, where $k'$ is an integer; therefore, $3n + 8$ is even.

ii) $Q(n) \rightarrow P(n)$

We will show instead its contrapositive, namely $\neg P(n) \rightarrow \neg Q(n)$, where $\neg P(n) : n$ is odd, and $\neg Q(n) : 3n + 8$ is odd.

Hypothesis: $n$ is odd. By definition of even numbers, there exists and integer $k$ such that $n = 2k + 1$. Then,

$$3n + 8 = 6k + 3 + 8 = 2(3k + 5) + 1$$

Since $3k + 5$ is an integer, $3n + 8$ can be written in the form $2k' + 1$, where $k'$ is an integer; therefore, $3n + 8$ is odd.
e) **Prove that either** \(4 \times 10^{769} + 22\) **or** \(4 \times 10^{769} + 23\) **is not a perfect square. Is your proof constructive, or non-constructive?**

Let \(n = 4 \times 10^{769} + 22\). The two numbers are \(n\) and \(n + 1\).

**Proof by contradiction:** Let us suppose that both \(n\) and \(n + 1\) are perfect squares:

\[
\exists k \in \mathbb{Z}, k^2 = n \\
\exists l \in \mathbb{Z}, l^2 = n + 1
\]

Then

\[
l^2 = k^2 + 1 \\
(l - k)(l + k) = 1
\]

Since \(l\) and \(k\) are integers, there are only two cases:

- \(l - k = 1\) and \(l + k = 1\), i.e. \(l = 1\) and \(k = 0\). Then we would have \(k^2 = 0\), i.e. \(n = 0\): contradiction
- \(l - k = -1\) and \(l + k = -1\), i.e. \(l = -1\) and \(k = 0\). Again, contradiction.

We can conclude that the proposition is true.

**Exercise 3 (10 points)**

Let \(n\) be a natural number and let \(a_1, a_2, \ldots, a_n\) be a set of \(n\) real numbers. Prove that at least one of these numbers is greater than, or equal to the average of these numbers. What kind of proof did you use?

We use a proof by contradiction.

Suppose none of the real numbers \(a_1, a_2, \ldots, a_n\) is greater than or equal to the average of these numbers, denoted by \(\bar{a}\).

By definition

\[
\bar{a} = \frac{a_1 + a_2 + \ldots + a_n}{n}
\]

Our hypothesis is that:

\[
a_1 < \bar{a} \\
 a_2 < \bar{a} \\
... < \bar{a} \\
a_n < \bar{a}
\]

We sum up all these equations and get the following:

\[
a_1 + a_2 + \ldots + a_n < n \cdot \bar{a}
\]

Replacing \(\bar{a}\) in equation (9) by its value given in equation (4) we get:
\[ a_1 + a_2 + ... + a_n < a_1 + a_2 + ... + a_n \]

This is not possible: a number cannot be strictly smaller than itself: we have reached a contradiction. Therefore our hypothesis was wrong, and the original statement was correct.

**Extra Credit (10 points)**

*Use Exercise 3 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17. (Hint: you can use the result from exercise 3)*

Let \( a_1, a_2, ..., a_{10} \) be an arbitrary order of 10 positive integers from 1 to 10 being placed around a circle:

\[
\begin{array}{c}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7 \\
  a_8 \\
  a_9 \\
  a_{10}
\end{array}
\]

Since the ten numbers \( a \) correspond to the first 10 positive integers, we get:

\[ a_1 + a_2 + ... + a_{10} = 1 + 2 + ... + 10 = 55 \quad (1) \]

Notice that the \( a_1, a_2, ..., a_{10} \) are not necessarily in the order 1, 2, ..., 10. They do include however the ten integers from 1 to 10: these is why the sum is 55.

Let us now consider the different sums \( S_i \) of three consecutive sites around the circle. There
are 10 such sums:

\[ S_1 = a_1 + a_2 + a_3 \]
\[ S_2 = a_2 + a_3 + a_4 \]
\[ S_3 = a_3 + a_4 + a_5 \]
\[ S_4 = a_4 + a_5 + a_6 \]
\[ S_5 = a_5 + a_6 + a_7 \]
\[ S_6 = a_6 + a_7 + a_8 \]
\[ S_7 = a_7 + a_8 + a_9 \]
\[ S_8 = a_8 + a_9 + a_{10} \]
\[ S_9 = a_9 + a_{10} + a_1 \]
\[ S_{10} = a_{10} + a_1 + a_2 \]

We do not know the values of the individual sums \( S_i \); however, we can compute the sum of these numbers:

\[
S_1 + S_2 + ... + S_{10} = (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + ... + (a_{10} + a_1 + a_2) \\
= 3 * (a_1 + a_2 + ... + a_{10}) \\
= 3 * 55 \\
= 165 
\]

The average of \( S_1, S_2, ..., S_{10} \) is therefore:

\[
\overline{S} = \frac{S_1 + S_2 + ... + S_{10}}{10} \\
= \frac{165}{10} \\
= 16.5
\]

Based on the conclusion of Exercise 3, at least one of \( S_1, S_2, ..., S_{10} \) is greater to or equal to \( \overline{S} \), i.e., 16.5. Because \( S_1, S_2, ..., S_{10} \) are all integers, they cannot be equal to 16.5. Thus, at least one of \( S_1, S_2, ..., S_{10} \) is greater to or equal to 17.