Exercise 1:

Show that

\[ 1 + 3 + 5 + \cdots + 2n - 1 = n^2 \quad \text{for all } n \geq 1 \]

Let us define:

- LHS \( (n) \): \[ 1 + 3 + 5 + \cdots + 2n - 1 \]
- RHS \( (n) \): \( n^2 \)

**P(n):** \( \text{LHS} \ (n) = \text{RHS} \ (n) \)

I want to show \( P(n) \) is true for all \( n \geq 1 \)

**Basis step:** We show \( P(1) \) is true

\[ \text{LHS} \ (1) = 1 \]
\[ \text{RHS} \ (1) = 1^2 = 1 \]

Therefore \( \text{LHS} \ (1) = \text{RHS} \ (1) \): \( P(1) \) is true.
Inductive step:

\[ P(n) \Rightarrow P(n+1) \quad \text{for all } n \geq 1 \]

We assume \( P(m) \) is true:

I know \( \text{LHS}(m) = \text{RHS}(m) \)

I need to show \( P(n+1) \) is true, meaning:

I need to show \( \text{LHS}(n+1) = \text{RHS}(n+1) \)

\[
\text{LHS}(n+1) = 1 + 3 + 5 + \cdots + 2n - 1 + 2n + 1
\]

\[ = \text{LHS}(m) + 2m + 1 \]

\[ = \text{RHS}(m) + 2m + 1 \]

\[ = m^2 + 2m + 1 \]

\[ = (m+1)^2 \]

\[
\text{RHS}(n+1) = (n+1)^2
\]

Therefore \( \text{LHS}(n+1) = \text{RHS}(n+1) \): \( P(n+1) \) is true.

The inductive step is true.

The method of proof by induction allows us to conclude that \( P(n) \) is true, \( \forall n \geq 1 \).
Exercise 2:
Show that \[
\sum_{i=1}^{m} \frac{1}{i(i+1)} = \frac{M}{m+1} \quad \forall m \geq 1
\]
\[
\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{m(m+1)} = \frac{M}{m+1} \quad \forall m \geq 1
\]

Let us define

LHS \( (m) = \frac{1}{2} + \cdot \cdots + \frac{1}{m(m+1)} \)

RHS \( (m) = \frac{M}{m+1} \)

\( P(m) : \) LHS \( (m) = \) RHS \( (m) \)

I want to show \( P(m) \) is true, \( \forall m \geq 1 \)

Basis step: I want to show \( P(1) \) is true

LHS \( (1) = \frac{1}{2} \)

RHS \( (1) = \frac{1}{1+1} = \frac{1}{2} \)

LHS \( (1) = \) RHS \( (1) \) : \( P(1) \) is true.
Inductive step:

\[ P(n) \rightarrow P(n+1) \quad \text{for all } n \geq 1 \]

We assume \( P(n) \) is true.

\[ \text{LHS}(n) = \text{RHS}(n) \]

I want to prove \( P(n+1) \) is true.

\[ \text{LHS}(n+1) = \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \]

\[ = \text{LHS}(n) + \frac{1}{(n+1)(n+2)} \]

\[ = \text{RHS}(n) + \frac{1}{(n+1)(n+2)} \]

\[ \text{RHS}(n+1) = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \]

\[ = \frac{n(n+2) + 1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} \]

Therefore \( \text{LHS}(n+1) = \text{RHS}(n+1) \): The inductive step is true.

The method of proof by induction allows me to conclude \( \sqrt{\text{that } P(n) \text{ is true for all } n} \).
Show that
\[ 1 + \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \quad \forall n \geq 1 \]

LHS(n) = 1 + \ldots + n^3
RHS(n) = \left( \frac{n(n+1)}{2} \right)^2

P(n): LHS(n) = RHS(n)

I want to prove P(n) is true \( \forall n \geq 1 \).

**Basis step:** We prove P(1) is true.

LHS(1) = 1
RHS(1) = \left( \frac{1(1+1)}{2} \right)^2 = 1

LHS(1) = RHS(1): P(1) is true.

**Induction step:** I need to show P(n) \( \Rightarrow \) P(n+1), i.e.

I assume P(n) is true.

Therefore LHS(n) = RHS(n)
\[ \text{LHS}(n+1) = 1 + \ldots + m^3 + (n+1)^3 \]
\[ = \text{LHS}(n) + (n+1)^3 \]
\[ = \text{RHS}(n) + (n+1)^3 \]
\[ = \left( \frac{m \cdot (m+1)}{2} \right)^2 + (n+1)^3 \]
\[ = \frac{m^2 \cdot (m-1)^2 + 4 \cdot (n+1)^3}{4} = \frac{(n+1)^2(n+2)^2}{4} \]

\[ \text{RHS}(n+1) = \left( \frac{(n+1)(n+2)}{2} \right)^2 \]

Therefore \( \text{LHS}(n+1) = \text{RHS}(n+1) \); the inductive step is true.

The method of proof by induction allows me to conclude that \( \sqrt{P(n)} \) is true for all \( n \geq 1 \).