**Induction**

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**Analogy:**

**Dominoes:**

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There are two conditions for all dominoes to fall:

1) If domino \( n \) falls, domino \( (n+1) \) falls.

2) You initiate the process at a domino \( m_0 \), usually \( m_0 = 1 \).

This is an actual method of proof called proof by induction.
Definition:

To prove that a proposition $P(n)$ is true for all $n \geq n_0$, $n \in \mathbb{N}$, it is enough to prove that:

a) basis step: $P(n_0)$ is true.

b) inductive step: $P(k) \Rightarrow P(k+1)$, for all $k \geq n_0$.

Example 1: Let $n$ be a natural number. How that:

$$\forall n \geq 1, \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \left(1 + 2 + \ldots + n\right)$$

Let us use the proof by induction. We define:

LHS($n$) = $1 + 2 + \ldots + n$

RHS($n$) = $\frac{n(n+1)}{2}$
I need to prove

∀ n ≥ 1, P(n) is true

where P(n): LHS(n) = RHS(n)

Basis step: n₀ = 1

LHS(1) = 1

RHS(1) = \frac{1(1+1)}{2} = 1

Therefore LHS(1) = RHS(1); P(1) is true.

Inductive step: I want to show P(k) → P(k+1)

when k ≥ (n₀ = 1)

p: LHS(k) = RHS(k)

q: LHS(k+1) = RHS(k+1)

I assume p is true: LHS(k) = RHS(k).

LHS(k+1) = \frac{1 + \ldots + k + k+1}{2}

= LHS(k) + k + 1

= RHS(k) + k + 1

= \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}
I found \[ \text{LHS} (k+1) = \frac{(k+1)(k+2)}{2} \]

\[ \text{RHS} (k+1) = \frac{(k+1)(k+2)}{2} \]

Therefore \[ \text{LHS} (k+1) = \text{RHS} (k+1) \]: \( q \) is true.

The inductive step is true.

The method of proof by induction allows us to conclude that \( \forall n \geq 1 \) \( \sqrt{P(n)} \) is true.

**Danger:**
Let us prove that all hoses have the same color.

**Proof:**
\[ \forall n \geq 1 \]
\[ P(n) : \text{n hoses have the same color.} \]

I want to show \( P(n) \) is true \( \forall n \geq 1 \).
Basis step:
\[ P(1) \text{ is true} \]

Inductive step:
\[ P(k) \rightarrow P(k+1) \text{ is true} \]

\[ P(k): \text{ } k \text{ horses have the same color.} \]

I assume \( P(k) \text{ is true} \).

\[ S = \{ h_1, h_2, \ldots, h_{k+1} \} \]

\( (k+1) \) horses.

\[ S_1 = \{ h_1, \ldots, h_k \} \rightarrow C_1 \]

\[ S_2 = \{ h_2, \ldots, h_{k+1} \} \rightarrow C_2 \]

\[ S_1 \cap S_2 = \{ h_2, \ldots, h_k \} \rightarrow \text{those horses have the color } C_1 \text{ and } C_2, \text{ therefore } C_1 = C_2. \]

Therefore \( P(k+1) \text{ is true} \).
The method of proof by induction allows me to conclude that all horses have the same color.