Exercise 1

Let $a$ and $b$ be two real numbers.

a) Show that $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$

Let $LHS = (a^2 + b^2)^2$ and $RHS = (a^2 - b^2)^2 + (2ab)^2$. Then:

$LHS = a^4 + b^4 + 2a^2b^2$

and

$RHS = a^4 + b^4 - 2a^2b^2 + 4a^2b^2$

$= a^4 + b^4 + 2a^2b^2$

Therefore $LHS = RHS$ for all $a$ and $b$, and the identity is true.

b) $a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$

Let $LHS = a^4 - b^4$ and $RHS = (a - b)(a + b)(a^2 + b^2)$. Then:

$RHS = (a^2 - b^2)(a^2 + b^2)$

$= a^4 - b^4$

Therefore $LHS = RHS$ for all $a, b$, and the identity is true.
Exercise 2

a) Show that there are no positive integer number \( n \) such that \( n^2 + n^3 = 100 \)

Let \( n \) be a positive integer. Since \( n > 0, n^2 > 0 \) and \( n^3 > 0 \). We note first that if \( n \geq 5 \), then \( n^3 \geq 125 \), i.e. \( n^3 > 100 \), and therefore \( n^2 + n^3 > 100 \). The only possible solutions are therefore \( n = 0, n = 1, n = 2, n = 3, \) and \( n = 4 \). We test each of those values separately:

i) \( n = 0 \) then \( n^2 + n^3 = 0 \neq 100. \) \( n = 0 \) is not a solution.

ii) \( n = 1 \) then \( n^2 + n^3 = 2 \neq 100. \) \( n = 1 \) is not a solution.

iii) \( n = 2 \) then \( n^2 + n^3 = 12 \neq 100. \) \( n = 2 \) is not a solution.

iv) \( n = 3 \) then \( n^2 + n^3 = 36 \neq 100. \) \( n = 3 \) is not a solution.

v) \( n = 4 \) then \( n^2 + n^3 = 80 \neq 100. \) \( n = 4 \) is not a solution.

Therefore there are no positive integer number \( n \) such that \( n^2 + n^3 = 100 \).

b) Prove that there are no solutions in integers \( x \) and \( y \) to the equation \( 2x^2 + 5y^2 = 14 \).

Let \( x \) and \( y \) be two integers. We note first that \( x^2 \geq 0 \) and \( y^2 \geq 0 \). Then, if \( y \leq -2 \) or \( y \geq 2 \), \( y^2 \geq 4 \) and \( 5y^2 \geq 20 \). Therefore we can conclude that \( y = -1, y = 0, \) or \( y = 1 \). We look at all three cases separately:

i) \( y = -1 \) or \( y = 1 \); then \( 2x^2 = 9 \); the left hand side is even, while the right hand side is odd: this equation has no solution.

ii) \( y = 0 \) then \( 2x^2 = 14 \) or \( x^2 = 7 \). We check all possible values of \( x \):

* \( x = 0 \); then \( x^2 = 0 \) → No.
* \( x = 1 \) or \( x = -1 \); then \( x^2 = 1 \) → No.
* \( x = 2 \) or \( x = -2 \); then \( x^2 = 4 \) → No.
* \( x \geq 3 \) or \( x < leq -3 \) then \( x^2 \geq 9 \) → No.

Therefore there are no integers \( x \) and \( y \) that satisfy the equation \( 2x^2 + 5y^2 = 14 \).

Exercise 3

Let \( x \) be a real number. Solve \( \sqrt{x^2 - 7} = \sqrt{1 - x^2} \)

We need to define the domain of the equation first. This equation involves two square root functions that are defined if and only if their arguments are positive. Therefore: \( D = \{ x \in \mathbb{R} | x^2 - 7 \geq 0 \ and \ 1 - x^2 \geq 0 \} \).

Let us look at both conditions:

i) \( x^2 - 7 \geq 0 \) implies that \( x \leq \sqrt{7} \) or \( x \geq \sqrt{7} \)

ii) \( 1 - x^2 \geq 0 \) implies that \( -1 \leq x \leq 1 \)

These two conditions are incompatible. Therefore \( D = \emptyset \), and the equation does not have any solutions.
Exercise 4

Three consecutive integers add up to 51. What are those three integers?

Let $a$ be an integer. The two integers that follow $a$ are $a + 1$ and $a + 2$. Therefore:

$$a + a + 1 + a + 2 = 51$$
$$3a + 3 = 51$$
$$3a = 48$$
$$a = 16$$

Therefore the three consecutive integers that add up to 51 are 16, 17, and 18.