Exercise 1: proofs

• a) Let $x$ and $y$ be two integers. Show that if $2x + 5y = 14$ and $y \neq 2$, then $x \neq 2$.

We need to prove an implication of the form $p \rightarrow q$, where $p$ and $q$ are defined as:

\[ p : 2x + 5y = 14 \text{ and } y \neq 2 \]
\[ q : x \neq 2 \]

We will use a proof by contradiction, namely we will suppose that the property is false, and find that this leads to a contradiction.

Hypothesis: $p \rightarrow q$ is false, which is equivalent to saying that $p$ is true, AND $q$ is false.

Therefore, $2x + 5y = 14$ and $y \neq 2$ and $x = 2$. Replacing $x$ by its value in the first equation, we get $4 + 5y = 14$, namely $y = 2$. Therefore we have $y = 2$ and $y \neq 2$: we have reached a contradiction.

Therefore the hypothesis is false, which means that $p \rightarrow q$ is true.

• b) Let $x$ and $y$ be two integers. Show that if $x^2 + y^2$ is odd, then $x + y$ is odd.

We need to prove an implication of the form $p \rightarrow q$, where $p$ and $q$ are defined as:

\[ p : x^2 + y^2 \text{ is odd} \]
\[ q : x + y \text{ is odd} \]

We will use an indirect proof, namely instead of showing that $p \rightarrow q$, we will show the equivalent property $\neg q \not\rightarrow \neg p$, where:

\[ \neg q : x + y \text{ is even} \]
\[ \neg p : x^2 + y^2 \text{ is even} \]

Hypothesis: $\neg q$ is true, namely $x + y$ is even. Since $x + y$ is even, $(x + y)^2$ is even (result from class). Therefore there exists an integer $k$ such that $(x + y)^2 = 2k$. We note also that:

\[ (x + y)^2 = x^2 + y^2 + 2xy, \]

Therefore,

\[ x^2 + y^2 = 2k - 2xy = 2(k - xy) \]

Since $k - xy$ is an integer, we conclude that $x^2 + y^2$ is even, namely that $\neg p$ is true.

We have shown that $\neg q \not\rightarrow \neg p$ is true; we can conclude that $p \rightarrow q$ is true.
Exercise 2
Let \( S = \{-1, 0, 2, 4, 7\} \). Find \( f(S) \) if:

- **a)**. \( f(x) = 1 \)
  
  Since \( f(x) = 1 \) for all elements of \( S \), \( f(S) = \{1\} \).

- **b)**. \( f(x) = 2x + 1 \)
  
  \( f(-1) = -1, f(0) = 1, f(2) = 5, f(4) = 9, \) and \( f(7) = 15 \). Therefore \( f(S) = \{-1, 1, 5, 9, 15\} \).

- **c)**. \( f(x) = \left\lfloor \frac{x}{5} \right\rfloor \)
  
  \( f(-1) = -1, f(0) = 0, f(2) = 0, f(4) = 0, \) and \( f(7) = 2 \). Therefore \( f(S) = \{-1, 0, 1\} \).

- **d)**. \( f(x) = \left\lfloor \frac{x^2+1}{3} \right\rfloor \)
  
  \( f(-1) = 0, f(0) = 0, f(2) = 1, f(4) = 5, \) and \( f(7) = 16 \). Therefore \( f(S) = \{0, 1, 5, 16\} \).

Exercise 3
Let \( S \) be a subset of a universe \( U \). The characteristic function \( f_S \) of \( S \) is the function from \( U \) to the set \( \{0, 1\} \) such that \( f_S(x) = 1 \) if \( x \) belongs to \( S \) and \( f_S(x) = 0 \) if \( x \) does not belong to \( S \). Let \( A \) and \( B \) be two sets. Show that for all \( x \) in \( U \),

- **a)**. \( f_{A \cap B}(x) = f_A(x)f_B(x) \)
  
  Let \( x \) be an element of \( U \). Let us call \( LHS(x) = f_{A \cap B}(x) \) and \( RHS(x) = f_A(x)f_B(x) \). We distinguish two cases:
  
  (i) \( x \in A \cap B \). Then \( LHS(x) = f_{A \cap B}(x) = 1 \), by definition of \( f_{A \cap B} \). Also, since \( x \in A \cap B \), then \( x \in A \) and \( x \in B \), therefore \( f_A(x) = 1 \) and \( f_B(x) = 1 \), i.e. \( RHS(x) = 1 \).
  
  (ii) \( x \notin A \cap B \). Then \( LHS(x) = f_{A \cap B}(x) = 0 \), by definition of \( f_{A \cap B} \). Also, since \( x \notin A \cap B \), then \( x \notin A \) or \( x \notin B \), therefore \( f_A(x) = 0 \) or \( f_B(x) = 0 \), i.e. \( RHS(x) = 0 \).
  
  The property is therefore true for all \( x \) in \( U \).

- **b)**. \( f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x)f_B(x) \)
  
  Let \( x \) be an element of \( U \). Let us call \( LHS(x) = f_{A \cup B}(x) \) and \( RHS(x) = f_A(x) + f_B(x) - f_A(x)f_B(x) \). We distinguish four cases:
  
  (i) \( x \in A \) and \( x \in B \). Then \( LHS(x) = f_{A \cap B}(x) = 1 \), as \( x \in A \cup B \). Also, \( f_A(x) = 1 \) and \( f_B(x) = 1 \), therefore \( RHS(x) = 1 + 1 - 1 = 1 \).
  
  (ii) \( x \in A \) and \( x \notin B \). Then \( LHS(x) = f_{A \cap B}(x) = 1 \), as \( x \in A \cup B \). Also, \( f_A(x) = 1 \) and \( f_B(x) = 0 \), therefore \( RHS(x) = 1 + 0 - 0 = 1 \).
  
  (iii) \( x \notin A \) and \( x \in B \). Then \( LHS(x) = f_{A \cap B}(x) = 1 \), as \( x \in A \cup B \). Also, \( f_A(x) = 0 \) and \( f_B(x) = 1 \), therefore \( RHS(x) = 0 + 1 - 0 = 1 \).
  
  (iv) \( x \notin A \) and \( x \notin B \). Then \( LHS(x) = f_{A \cap B}(x) = 0 \), as \( x \notin A \cup B \). Also, \( f_A(x) = 0 \) and \( f_B(x) = 0 \), therefore \( RHS(x) = 0 + 0 - 0 = 0 \).
  
  The property is therefore true for all \( x \) in \( U \).
Exercise 4

Let \( n \) be an integer. Show that \( \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor \).

Let \( n \) be an integer. We define \( \text{LHS}(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \) and \( \text{RHS}(n) = \left\lfloor \frac{n^2}{4} \right\rfloor \). Since we consider the division of \( n \) by 2, we consider two cases:

(i) \( n \) is even. Then there exists \( k \in \mathbb{Z} \) such that \( n = 2k \). Then:

\[
\left\lfloor \frac{n}{2} \right\rfloor = k, \quad \left\lceil \frac{n}{2} \right\rceil = k, \text{ therefore } \text{LHS}(n) = k^2.
\]

\( n^2 = 4k^2 \), therefore \( \left\lfloor \frac{n^2}{4} \right\rfloor = k^2 \), i.e. \( \text{RHS}(n) = k^2 \).

(ii) \( n \) is odd. Then there exists \( k \in \mathbb{Z} \) such that \( n = 2k + 1 \). Then:

\[
\left\lfloor \frac{n}{2} \right\rfloor = k, \quad \left\lceil \frac{n}{2} \right\rceil = k + 1, \text{ therefore } \text{LHS}(n) = k^2 + k.
\]

\( n^2 = 4k^2 + 4k + 1 \), therefore \( \left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k \), i.e. \( \text{RHS}(n) = k^2 + k \).

The property is therefore true for all \( n \) in \( \mathbb{Z} \).