Exercise 0: Additional problems on proofs

- **a)** Let \( x \) and \( y \) be two integers. Show that if \( 2x + 5y = 14 \) and \( y \neq 2 \), then \( x \neq 2 \).

  We need to prove an implication of the form \( p \rightarrow q \), where \( p \) and \( q \) are defined as:
  
  \[
  p : 2x + 5y = 14 \text{ and } y \neq 2
  \]
  
  \[
  q : x \neq 2
  \]

  We will use a proof by contradiction, namely we will suppose that the property is false, and find that this leads to a contradiction.

  Hypothesis: \( p \rightarrow q \) is false, which is equivalent to saying that \( p \) is true, AND \( q \) is false.

  Therefore, \( 2x + 5y = 14 \) and \( y \neq 2 \) and \( x = 2 \). Replacing \( x \) by its value in the first equation, we get \( 4 + 5y = 14 \), namely \( y = 2 \). Therefore we have \( y = 2 \) and \( y \neq 2 \): we have reached a contradiction.

  Therefore the hypothesis is false, which means that \( p \rightarrow q \) is true.

- **b)** Let \( x \) and \( y \) be two integers. Show that if \( x^2 + y^2 \) is odd, then \( x + y \) is odd.

  We need to prove an implication of the form \( p \rightarrow q \), where \( p \) and \( q \) are defined as:
  
  \[
  p : x^2 + y^2 \text{ is odd}
  \]
  
  \[
  q : x + y \text{ is odd}
  \]

  We will use an indirect proof, namely instead of showing that \( p \rightarrow q \), we will show the equivalent property \( \lnot q \rightarrow \lnot p \), where:

  \[
  \lnot q : x + y \text{ is even}
  \]
  
  \[
  \lnot p : x^2 + y^2 \text{ is even}
  \]

  Hypothesis: \( \lnot q \) is true, namely \( x + y \) is even. Since \( x + y \) is even, \( (x + y)^2 \) is even (result from class). Therefore there exists an integer \( k \) such that \( (x + y)^2 = 2k \). We note also that:

  \[
  (x + y)^2 = x^2 + y^2 + 2xy,
  \]

  Therefore,

  \[
  x^2 + y^2 = 2k - 2xy = 2(k - xy)
  \]

  Since \( k - xy \) is an integer, we conclude that \( x^2 + y^2 \) is even, namely that \( \lnot p \) is true.

  We have shown that \( \lnot q \rightarrow \lnot p \) is true; we can conclude that \( p \rightarrow q \) is true.
Exercise 1

To show that \( f \) is bijective (or not) from \( \mathbb{R} \) to \( \mathbb{R} \), we need to check: (i) that it is a function, (ii) that it is one-to-one (injective), and (iii) that it is onto (surjective).

- **a)** \( f(x) = 2x + 4 \)
  
  (i) \( f \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \), as its domain is \( \mathbb{R} \)
  
  (ii) Let us show that \( f \) is injective. Let \( x \) and \( y \) be two real numbers such that \( f(x) = f(y) \). Then \( 2x + 4 = 2y + 4 \), therefore \( x = y \). \( f \) is injective.
  
  (iii) Let us show that \( f \) is surjective. Let \( y \) be an element of the co-domain, \( \mathbb{R} \). To find if there exists a real number \( x \) such that \( f(x) = y \), we solve the equation \( f(x) = y \), i.e. \( 2x + 4 = y \). We find \( x = \frac{y - 4}{2} \), i.e. \( x \) exists for each value of \( y \). \( f \) is surjective.

  We conclude that \( f \) is bijective.

- **b)** \( f(x) = x^2 + 1 \)
  
  (i) \( f \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \), as its domain is \( \mathbb{R} \)
  
  (ii) Is \( f \) injective?. Let \( x \) and \( y \) be two real numbers such that \( f(x) = f(y) \). Then \( x^2 + 1 = y^2 + 1 \), i.e. \( x^2 - y^2 = 0 \). This leads to \( (x - y)(x + y) = 0 \), therefore \( x = y \) or \( x = -y \). For example, \( f(1) = f(-1) \): \( f \) is not injective; it is therefore not bijective.

- **c)** \( f(x) = \frac{(x + 1)}{(x + 2)} \)
  
  (i) \( f \) is not a function from \( \mathbb{R} \) to \( \mathbb{R} \), as it is not defined for \( x = -2 \). The domain \( D \) is \( \mathbb{R} - \{-2\} \).

  It is a function from \( D \) to \( \mathbb{R} \). Is it a bijection from \( D \) to \( \mathbb{R} \)?
  
  (ii) Let \( x \) and \( y \) be two real numbers such that \( f(x) = f(y) \). Then \( (x + 1)/(x + 2) = (y + 1)/(y + 2) \), i.e. \( (x + 1)(y + 2) = (y + 1)(x + 2) \). After development, we get \( 2x + y = 2y + x \) i.e. \( x = y \). The function is injective.

  (iii) Let \( y \) be an element of the co-domain, \( \mathbb{R} \). To find if there exists a real number \( x \) such that \( f(x) = y \), we solve the equation \( f(x) = y \), i.e. \( (x + 1)/(x + 2) = y \). This becomes \( x + 1 = y(x + 2), \) i.e. \( x(1 - y) = 2y - 1 \), which has a solution if and only if \( y \neq 1 \). Therefore we found one element of the co-domain \( (y = 1) \) for which we cannot find an element \( x \) such that \( f(x) = y \). \( f \) is not surjective, therefore \( f \) is not bijective.

- **d)** \( f(x) = \frac{(x^2 + 1)}{(x^2 + 2)} \)
  
  (i) \( f \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \), as its domain is \( \mathbb{R} \)
  
  (ii) Is \( f \) injective? We note that \( f(1) = f(-1) \): \( f \) is not injective, therefore \( f \) is not bijective.

Exercise 2

Let \( S = \{-1, 0, 2, 4, 7\} \). Find \( f(S) \) if:

- **a).** \( f(x) = 1 \)

  Since \( f(x) = 1 \) for all elements of \( S \), \( f(S) = \{1\} \).

- **b).** \( f(x) = 2x + 1 \)

  \( f(-1) = -1, f(0) = 1, f(2) = 5, f(4) = 9, \) and \( f(7) = 15 \). Therefore \( f(S) = \{-1, 1, 5, 9, 15\} \).
• c). \( f(x) = \left\lfloor \frac{x}{2} \right\rfloor \)
\( f(-1) = -1, f(0) = 0, f(2) = 0, f(4) = 0, \) and \( f(7) = 2. \) Therefore \( f(S) = \{-1, 0, 1\}. \)

• d). \( f(x) = \left\lfloor \frac{x^2+1}{3} \right\rfloor \)
\( f(-1) = 0, f(0) = 0, f(2) = 1, f(4) = 5, \) and \( f(7) = 16. \) Therefore \( f(S) = \{0, 1, 5, 16\}. \)

### Exercise 3

Let \( S \) be a subset of a universe \( U. \) The characteristic function \( f_S \) of \( S \) is the function from \( U \) to the set \( \{0, 1\} \) such that \( f_S(x) = 1 \) if \( x \) belongs to \( S \) and \( f_S(x) = 0 \) if \( x \) does not belong to \( S. \) Let \( A \) and \( B \) be two sets. Show that for all \( x \) in \( U, \)

- **a).** \( f_A \cap B(x) = f_A(x) f_B(x) \)

  Let \( x \) be an element of \( U. \) We distinguish two cases:
  
  (i) \( x \in A \cap B. \) Then \( LHS(x) = f_A \cap B(x) = 1, \) by definition of \( f_A \cap B. \) Also, since \( x \in A \cap B, \) then \( x \in A \) and \( x \in B, \) therefore \( f_A(x) = 1 \) and \( f_B(x) = 1, \) i.e. \( RHS(x) = 1. \)
  
  (ii) \( x \notin A \cap B. \) Then \( LHS(x) = f_A \cap B(x) = 0, \) by definition of \( f_A \cap B. \) Also, since \( x \notin A \cap B, \) then \( x \notin A \) or \( x \notin B, \) therefore \( f_A(x) = 0 \) or \( f_B(x) = 0, \) i.e. \( RHS(x) = 0. \)

  The property is therefore true for all \( x \) in \( U. \)

- **b).** \( f_A \cup B(x) = f_A(x) + f_B(x) - f_A(x)f_B(x) \)

  Let \( x \) be an element of \( U. \) We distinguish four cases:
  
  (i) \( x \in A \) and \( x \in B. \) Then \( LHS(x) = f_A \cup B(x) = 1, \) as \( x \in A \cup B. \) Also, \( f_A(x) = 1 \) and \( f_B(x) = 1, \) therefore \( RHS(x) = 1 + 1 - 1 = 1. \)
  
  (ii) \( x \in A \) and \( x \notin B. \) Then \( LHS(x) = f_A \cup B(x) = 1, \) as \( x \in A \cup B. \) Also, \( f_A(x) = 1 \) and \( f_B(x) = 0, \) therefore \( RHS(x) = 1 + 0 - 0 = 1. \)
  
  (iii) \( x \notin A \) and \( x \in B. \) Then \( LHS(x) = f_A \cup B(x) = 1, \) as \( x \in A \cup B. \) Also, \( f_A(x) = 0 \) and \( f_B(x) = 1, \) therefore \( RHS(x) = 0 + 1 - 0 = 1. \)
  
  (iv) \( x \notin A \) and \( x \notin B. \) Then \( LHS(x) = f_A \cup B(x) = 0, \) as \( x \notin A \cup B. \) Also, \( f_A(x) = 0 \) and \( f_B(x) = 0, \) therefore \( RHS(x) = 0 + 0 - 0 = 0. \)

  The property is therefore true for all \( x \) in \( U. \)

### Exercise 4

Let \( n \) be an integer. Show that \( \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor. \)

Let \( n \) be an integer. We define \( LHS(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \) and \( RHS(n) = \left\lfloor \frac{n^2}{4} \right\rfloor. \) Since we consider the division of \( n \) by \( 2, \) we consider two cases:

(i) \( n \) is even. Then there exists \( k \in \mathbb{Z} \) such that \( n = 2k. \) Then:

\( \left\lfloor \frac{n}{2} \right\rfloor = k, \) \( \left\lfloor \frac{n}{2} \right\rfloor = k, \) \( \) therefore \( LHS(n) = k^2. \)

\( n^2 = 4k^2, \) therefore \( \left\lfloor \frac{n^2}{4} \right\rfloor = k^2, \) i.e. \( RHS(n) = k^2. \)

(ii) \( n \) is odd. Then there exists \( k \in \mathbb{Z} \) such that \( n = 2k + 1. \) Then:

\( \frac{n}{2} = k + \frac{1}{2}. \) Then, \( \left\lfloor \frac{n}{2} \right\rfloor = k, \) \( \left\lfloor \frac{n}{2} \right\rfloor = k + 1, \) therefore \( LHS(n) = k^2 + k. \)
\[ n^2 = 4k^2 + 4k + 1, \text{ therefore } \left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k, \text{ i.e. } RHS(n) = k^2 + k. \]
The property is therefore true for all \( n \) in \( \mathbb{Z} \).