Exercise 1: proofs

• a) Let $x$ and $y$ be two integers. Show that if $2x + 5y = 14$ and $y \neq 2$, then $x \neq 2$.

We need to prove an implication of the form $p \rightarrow q$, where $p$ and $q$ are defined as:

\begin{align*}
p & : 2x + 5y = 14 \text{ and } y \neq 2 \\
q & : x \neq 2
\end{align*}

We will use a proof by contradiction, namely we will suppose that the property is false, and find that this leads to a contradiction.

Hypothesis: $p \rightarrow q$ is false, which is equivalent to saying that $p$ is true, AND $q$ is false.

Therefore, $2x + 5y = 14$ and $y \neq 2$ and $x = 2$. Replacing $x$ by its value in the first equation, we get $4 + 5y = 14$, namely $y = 2$. Therefore we have $y = 2$ and $y \neq 2$: we have reached a contradiction.

Therefore the hypothesis is false, which means that $p \rightarrow q$ is true.

• b) Let $x$ and $y$ be two integers. Show that if $x^2 + y^2$ is odd, then $x + y$ is odd.

We need to prove an implication of the form $p \rightarrow q$, where $p$ and $q$ are defined as:

\begin{align*}
p & : x^2 + y^2 \text{ is odd} \\
q & : x + y \text{ is odd}
\end{align*}

We will use an indirect proof, namely instead of showing that $p \rightarrow q$, we will show the equivalent property $\neg q \rightarrow \neg p$, where:

\begin{align*}
\neg q & : x + y \text{ is even} \\
\neg p & : x^2 + y^2 \text{ is even}
\end{align*}

Hypothesis: $\neg q$ is true, namely $x + y$ is even. Since $x + y$ is even, $(x + y)^2$ is even (result from class). Therefore there exists an integer $k$ such that $(x + y)^2 = 2k$. We note also that:

\[(x + y)^2 = x^2 + y^2 + 2xy,\]

Therefore,

\[x^2 + y^2 = 2k - 2xy = 2(k - xy)\]

Since $k - xy$ is an integer, we conclude that $x^2 + y^2$ is even, namely that $\neg p$ is true.

We have shown that $\neg q \rightarrow \neg p$ is true; we can conclude that $p \rightarrow q$ is true.
Exercise 2: floor and ceiling

• a). Let $x$ be a real number. Show that:

$$\left\lfloor \left\lfloor \frac{x}{2} \right\rfloor \right\rfloor = \left\lfloor \frac{x}{4} \right\rfloor$$

Let us define $k = \left\lfloor \frac{x}{2} \right\rfloor$ and $m = \left\lfloor \frac{x}{4} \right\rfloor$. By definition of floor, we have the two properties:

$k \leq \frac{x}{2} < k + 1$

and

$m \leq \frac{x}{4} < m + 1$

Let us multiply the second inequalities by 2:

$2m \leq \frac{x}{2} < 2(m + 1)$

We notice that:

$k \leq \frac{x}{2}$ and $\frac{x}{2} < 2(m + 1)$; therefore $k < 2(m + 1)$.

$k \leq \frac{x}{2}$ and $2m \leq \frac{x}{2}$. Therefore $k$ and $2m$ are two integers smaller than $\frac{x}{2}$. By definition of floor, $k$ is the largest integer smaller that $\frac{x}{2}$. Therefore $2m < k$.

Combining those two inequalities, we get $2m \leq k < 2(m + 1)$. After division by 2, $m < \frac{k}{2} < m + 1$. Therefore $m$ is the floor of $\frac{k}{2}$. Replacing $m$ and $k$ by their values, we get:

$$m = \left\lfloor \frac{x}{4} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \left\lfloor \frac{x}{2} \right\rfloor \right\rfloor$$

The property is therefore true.

• b). Let $n$ be an odd integer. Show that

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2 + 3}{4}$$

We use a direct proof. As $n$ is an odd integer, there exists an integer $k$ such that $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1$. Therefore,

$LHS = \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor k^2 + k + \frac{1}{4} \right\rfloor = k^2 + k + \left\lfloor \frac{1}{4} \right\rfloor = k^2 + k + 1$

and

$RHS = \frac{n^2 + 3}{4} = \frac{4k^2 + 4k + 4}{4} = k^2 + k + 1$

Therefore $LHS = RHS$; the property is true.

Exercise 3

• a). Show that if a function $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ is $O(x)$, then $f(x)$ is $O(x^2)$.

By definition of $O$, there exists a real number $k$ and a constant $C$ such that if $x > k$, then $|f(x)| < C|x|$. Let

$k_2 = \max(k, 1)$. Since $k_2 > k$, we have that for $x > k_2$,
Since $k_2 > 1$, we have that for $x > k_2$,

$$|x| < |x^2|$$

Combining those two inequalities, we get that for $x > k_2$,

$$|f(x)| < C|x^2|$$

Therefore $f(x)$ is $O(x^2)$.

• b). Show that $f(n) = n \log(n^2 + 1) + \frac{\log(n)}{n^{5/2}+1}$ is $O(n \log(n))$.

Notice first that $f(n)$ can be written as the sum of two functions $g(n) = n \log(n^2 + 1)$ and $h(n) = \frac{\log(n)}{n^{5/2}+1}$. Let us work separately with $g(n)$ and $h(n)$:

i) Notice that:

$$g(n) = n \log(n^2(1 + \frac{1}{n^2})) = 2n \log(n) + n \log(1 + \frac{1}{n^2})$$

Since $\frac{1}{n^2} < 1$ for $n > 1$, $1 + \frac{1}{n^2} < 2$ and $n \log(1 + \frac{1}{n^2}) < n \log(2)$. Therefore $n \log(1 + \frac{1}{n^2})$ is $O(n)$. Since $2n \log(n)$ is $O(n \log(n))$, we conclude that $g(n)$ is $O(n \log(n))$.

ii) Notice that

$$h(n) = \frac{\log(n)}{n^{5/2}+1} < \frac{n}{n^{5/2}+1} < n$$

Therefore $h(n)$ is $O(n)$.

We found that $g(n)$ is $O(n \log(n))$ and $h(n)$ is $O(n)$: $f(n) = g(n) + h(n)$ is therefore $O(\max(n \log(n), n))$, namely $O(n \log(n))$. 

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