Exercise 1

Let $a$, $b$, and $c$ be three integers. Show that if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

We use a direct proof. Hypothesis: $a \mid bc$ and $\gcd(a, b) = 1$

Since $\gcd(a, b) = 1$, according to Bezout’s identity, we know that there exists two integer numbers $m$ and $n$ such that

$$am + bn = 1$$

After multiplication by $c$:

$$acm + bcn = c$$

We know that $a \mid bc$. Therefore there exists an integer $k$ such that $bc = ka$. Replacing in the equation above, we get:

$$acm + kan = c$$

$$a(cm + kn) = c$$

i.e. $a \mid c$.

Exercise 2

Let $n$ be a natural number. We call $s(n)$ the sum of its digits. We want to show that if $s(3n) = s(n)$ then $9 \mid n$.

Proof. We use a direct proof.

Let $n$ be a natural number. Since $3 \mid 3n$, we know that $3 \mid s(3n)$ (this is the divisibility property: a number is divisible by 3 if and only if 3 divides the sum of its digit).

The hypothesis is that $s(3n) = s(n)$. Therefore $3 \mid s(n)$, i.e. $3 \mid n$ (from the same divisibility by 3 property).

As $3 \mid n$, there exists an integer $k$ such that $n = 3k$. Then $3n = 9k$, i.e. $9 \mid 3n$. Applying the divisibility by 9 property (i.e. a number is divisible by 9 if and only if 9 divides the sum of its digits), we find that $9 \mid s(3n)$. Therefore $9 \mid s(n)$ and finally $9 \mid n$. 
Exercise 3

Let a be a non-zero integer. Show that if $2 \nmid a$ and $3 \nmid a$, then $24 \mid (a^2 + 23)$.

Proof: we use a direct proof.

Let us consider the division of $a$ by 6: there exists $q$ and $r$ such that $a = 6q + r$, with $0 \leq r < 6$. We note that $r \neq 0$ and $r \neq 2$ and $r \neq 4$, as otherwise we would have $2 \mid a$. Similarly, $r \neq 3$, as otherwise $3 \mid a$. There are only two cases left: $r = 1$ or $r = 5$. We consider the two cases separately:

1) $r = 1$

$a = 6q + 1$, therefore $a^2 + 23 = (6q + 1)^2 + 23 = 36k^2 + 12k + 24 = 12k(3k + 1) + 24$. As $k$ is an integer, we consider two cases:

$k$ is even .

There exists an integer $l$ such that $k = 2l$. Therefore, $a^2 + 23 = 24l(3k + 1) + 24 = 24(l(3k + 1) + 1)$, i.e. $24 \mid (a^2 + 23)$.

$k$ is odd .

There exists an integer $l$ such that $k = 2l + 1$. Then $3k + 1 = 6l + 4 = 2(3l + 2)$. Therefore $a^2 + 23 = 24k(3l + 2) + 24 = 24(k(3l + 2) + 1)$, i.e. $24 \mid (a^2 + 23)$.

We can conclude that when $a = 6q + 1$, $24 \mid (a^2 + 23)$.

2) $r = 5$

$a = 6q + 5$, therefore $a^2 + 23 = (6q + 5)^2 + 23 = 36k^2 + 60k + 48 = 12k(3k + 5) + 48$. As $k$ is an integer, we consider two cases:

$k$ is even .

There exists an integer $l$ such that $k = 2l$. Therefore, $a^2 + 23 = 24l(3k + 5) + 48 = 24(l(3k + 5) + 2)$, i.e. $24 \mid (a^2 + 23)$.

$k$ is odd .

There exists an integer $l$ such that $k = 2l + 1$. Then $3k + 5 = 6l + 8 = 2(3l + 4)$. Therefore $a^2 + 23 = 24k(3l + 4) + 48 = 24(k(3l + 4) + 2)$, i.e. $24 \mid (a^2 + 23)$.

We can conclude that when $a = 6q + 1$, $24 \mid (a^2 + 23)$.

The property is therefore true for all $a$ such that $2 \nmid a$ and $3 \nmid a$.

Exercise 4

Since $x$, $y$, and $z$ are natural numbers greater than 1, the number $(xyz + 1)$ is not divisible by either $x$, $y$, or $z$, as $xyz$ is a multiple of all of the three numbers, and $(xyz + 1) \equiv 1 \mod x$, $(xyz + 1) \equiv 1 \mod y$ and $(xyz + 1) \equiv 1 \mod z$. Thus, we have proved by constructive proof that there exists at least one number greater than $x$, $y$, and $z$, which is not divisible by either of the three.