

**Exercise 1**

Let $a$, $b$, and $c$ be three integers. Show that if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

We use a direct proof. Hypothesis: $a \mid bc$ and $\gcd(a, b) = 1$.

Since $\gcd(a, b) = 1$, according to Bezout’s identity, we know that there exists two integer numbers $m$ and $n$ such that

$$am + bn = 1$$

After multiplication by $c$:

$$acm + bcn = c$$

We know that $a \mid bc$. Therefore there exists an integer $k$ such that $bc = ka$. Replacing in the equation above, we get:

$$acm + kan = c$$

$$a(cm + kn) = c$$

i.e. $a \mid c$.

**Exercise 2**

Let $n$ be a natural number. We call $s(n)$ the sum of its digits. We want to show that if $s(3n) = s(n)$ then $9 \mid n$.

Proof. We use a direct proof.

Let $n$ be a natural number. Since $3 \mid 3n$, we know that $3 \mid s(3n)$ (this is the divisibility property: a number is divisible by 3 if and only if 3 divides the sum of its digit).

The hypothesis is that $s(3n) = s(n)$. Therefore $3 \mid s(n)$, i.e. $3 \mid n$ (from the same divisibility by 3 property).

As $3 \mid n$, there exists an integer $k$ such that $n = 3k$. Then $3n = 9k$, i.e. $9 \mid 3n$. Applying the divisibility by 9 property (i.e. a number is divisible by 9 if and only if 9 divides the sum of its digits), we find that $9 \mid s(3n)$. Therefore $9 \mid s(n)$ and finally $9 \mid n$. 

Exercise 3

Let \( a \) be a non-zero integer. Show that if \( 2 \nmid a \) and \( 3 \nmid a \), then \( 24 \mid (a^2 + 23) \).

Proof: we use a direct proof.

Let us consider the division of \( a \) by 6: there exists \( q \) and \( r \) such that \( a = 6q + r \), with \( 0 \leq r < 6 \). We note that \( r \neq 0 \) and \( r \neq 2 \) and \( r \neq 4 \), as otherwise we would have \( 2 \mid a \). Similarly, \( r \neq 3 \), as otherwise \( 3 \mid a \). There are only two cases left: \( r = 1 \) or \( r = 5 \). We consider the two cases separately:

1) \( r = 1 \)

\( a = 6q + 1 \), therefore \( a^2 + 23 = (6q + 1)^2 + 23 = 36k^2 + 12k + 24 = 12k(3k + 1) + 24 \). As \( k \) is an integer, we consider two cases:

\( k \) is even .

There exists an integer \( l \) such that \( k = 2l \). Therefore, \( a^2 + 23 = 24l(3k + 1) + 24 = 24l(3k + 1) + 24 \), i.e. \( 24 \mid (a^2 + 23) \).

\( k \) is odd .

There exists an integer \( l \) such that \( k = 2l + 1 \). Then \( 3k + 1 = 6l + 4 = 2(3l + 2) \). Therefore \( a^2 + 23 = 24k(3l + 2) + 24 = 24(k(3l + 2) + 1) \), i.e. \( 24 \mid (a^2 + 23) \).

We can conclude that when \( a = 6q + 1 \), \( 24 \mid (a^2 + 23) \).

2) \( r = 5 \)

\( a = 6q + 5 \), therefore \( a^2 + 23 = (6q + 5)^2 + 23 = 36k^2 + 60k + 48 = 12k(3k + 5) + 48 \). As \( k \) is an integer, we consider two cases:

\( k \) is even .

There exists an integer \( l \) such that \( k = 2l \). Therefore, \( a^2 + 23 = 24l(3k + 5) + 48 = 24l(3k + 5) + 48 \), i.e. \( 24 \mid (a^2 + 23) \).

\( k \) is odd .

There exists an integer \( l \) such that \( k = 2l + 1 \). Then \( 3k + 5 = 6l + 8 = 2(3l + 4) \). Therefore \( a^2 + 23 = 24k(3l + 4) + 48 = 24(k(3l + 4) + 2) \), i.e. \( 24 \mid (a^2 + 23) \).

We can conclude that when \( a = 6q + 1 \), \( 24 \mid (a^2 + 23) \).

The property is therefore true for all \( a \) such that \( 2 \nmid a \) and \( 3 \nmid a \).

Exercise 4

Since \( x, y, \) and \( z \) are natural numbers greater than 1, the number \((xyz+1)\) is not divisible by either \( x, y \) or \( z \), as \( xyz \) is a multiple of all of the three numbers, and \((xyz+1) \equiv 1 \mod x, (xyz+1) \equiv 1 \mod y \) and \((xyz+1) \equiv 1 \mod z \). Thus, we have proved by constructive proof that there exists at least one number greater than \( x, y, \) and \( z \), which is not divisible by either of the three.