Induction

Exercise a

Let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n + 1)}{2}$$

We want to show that $P(n)$ is true for all $n > 0$. Let us define: $LHS(n) = \sum_{i=1}^{n} (-1)^i i^2$ and $RHS(n) = \frac{(-1)^n n(n + 1)}{2}$.

- **Basic step:**
  $$LHS(1) = (-1) \times 1^2 = 1 \quad \quad \quad RHS(1) = \frac{(-1) \times 1 \times 2}{2} = 1$$
  Therefore $P(1)$ is true.

- **Induction step:** We suppose that $P(k)$ is true, with $1 \leq k$. We want to show that $P(k + 1)$ is true.
\[ LHS(k + 1) = \sum_{i=1}^{k+1} (-1)^i \cdot i^2 \]

\[ = \sum_{i=1}^{k} (-1)^i \cdot i^2 + (-1)^{k+1} (k + 1)^2 \]

\[ = LHS(k) + (-1)^{k+1} (k + 1)^2 \]

\[ = RHS(k) + (-1)^{k+1} (k + 1)^2 \]

\[ = \frac{(-1)^k k(k + 1)}{2} + (-1)^{k+1} (k + 1)^2 \]

\[ = \frac{(-1)^k k(k + 1) + 2(-1)^{k+1} (k + 1)^2}{2} \]

\[ = \frac{(-1)^{k+1} (k + 1)(2k + 2 - k)}{2} \]

\[ \frac{(-1)^{k+1} (k + 1)(k + 2)}{2} \]

and

\[ RHS(k + 1) = \frac{(-1)^{k+1} (k + 1)(k + 2)}{2} \]

Therefore \( LHS(k + 1) = RHS(k + 1) \), which validates that \( P(k + 1) \) is true.

The principle of proof by mathematical induction allows us to conclude that \( P(n) \) is true for all \( n > 0 \).

**Exercise b**

Let \( P(n) \) be the proposition: \( 2^n \leq n! \). Let us define \( LHS(n) = 2^n \) and \( RHS(n) = n! \). We want to show that \( P(n) \) is true for all \( n \geq 4 \).

- **Basis step**: We show that \( P(4) \) is true:

\[ LHS(4) = 2^4 = 16 \]

\[ RHS(4) = 4! = 24 \]

Therefore \( LHS(4) \leq RHS(4) \) and \( P(4) \) is true.

- **Inductive step**: Let \( k \) be a positive integer greater or equal to 4 (\( k \geq 4 \)), and let us suppose that \( P(k) \) is true. We want to show that \( P(k + 1) \) is true.

\[ LHS(k + 1) = 2^{k+1} = 2LHS(k) \]

Since \( P(k) \) is true, we find:

\[ LHS(k + 1) \leq 2k! \]
Since $k \geq 4$, $2 \leq k + 1$.
Therefore

$$LHS(k + 1) \leq (k + 1) \times k!$$

$$LHS(k + 1) \leq (k + 1)!$$

Since $RHS(k + 1) = (k + 1)!$, we get $LHS(k + 1) < RHS(k + 1)$ which validates that $P(k + 1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 4$.

**Exercise c**

Let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} \frac{1}{i(i + 1)} = \frac{n}{n + 1}$$

We want to show that $P(n)$ is true for all $n > 0$. Let us define: $LHS(n) = \sum_{i=1}^{n} \frac{1}{i(i + 1)}$ and $RHS(n) = \frac{n}{n+1}$.

- **Basic step:**
  
  $$LHS(1) = \frac{1}{1 \times 2} = \frac{1}{2}, \quad RHS(1) = \frac{1}{2}$$

  Therefore $P(1)$, $P(2)$ and $P(3)$ are true.

- **Induction step:** We suppose that $P(k)$ is true, with $1 \leq k$. We want to show that $P(k + 1)$ is true.
\[ LHS(k + 1) = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} \]
\[ = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \]
\[ = LHS(k) + \frac{1}{(k+1)(k+2)} \]
\[ = RHS(k) + \frac{1}{(k+1)(k+2)} \]
\[ = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \]
\[ = k(k+2) + 1 \]
\[ = (k+1)(k+2) \]
\[ = k^2 + 2k + 1 \]
\[ = (k+1)^2 \]
\[ = \frac{(k+1)^2}{(k+1)(k+2)} \]
\[ = \frac{k+1}{k+2} \]

and

\[ RHS(k + 1) = \frac{k + 1}{k + 2} \]

Therefore \( LHS(k + 1) = RHS(k + 1) \), which validates that \( P(k + 1) \) is true.

The principle of proof by mathematical induction allows us to conclude that \( P(n) \) is true for all \( n \).

**Fibonacci**

**Exercise a**

Let \( P(n) \) be the proposition: \( f_1 + f_2 + \ldots + f_n = f_{n+2} - 1 \). We define \( LHS(n) = f_1 + f_2 + \ldots + f_n \) and \( RHS(n) = f_{n+2} - 1 \). We want to show that \( P(n) \) is true for all \( n \).

- **Basic step:**
  
  \[
  LHS(1) = f_1 = 1 \\
  RHS(1) = f_3 - 1 = 2 - 1 = 1 
  \]

  Therefore \( LHS(1) = RHS(1) \) and \( P(1) \) is true.

- **Inductive step:** Let \( k \) be a positive integer, and let us suppose that \( P(k) \) is true. We want to show that \( P(k + 1) \) is true.
Then

\[ \text{LHS}(k + 1) = f_1 + f_2 + \ldots + f_k + f_{k+1} \]
\[ = \text{LHS}(k) + f_{k+1} \]
\[ = \text{RHS}(k) + f_{k+1} \]
\[ = f_{k+2} - 1 + f_{k+1} \]
\[ = f_{k+1} + f_{k+2} - 1 \]
\[ = f_{k+3} - 1 \]

and

\[ \text{RHS}(k + 1) = f_{k+3} - 1 \]

Therefore \( \text{LHS}(k + 1) = \text{RHS}(k + 1) \), which validates that \( P(k + 1) \) is true.

The principle of proof by mathematical induction allows us to conclude that \( P(n) \) is true for all \( n \).

**Exercise b**

Let \( P(n) \) be the proposition: \( f_{4n} \) is divisible by 3. We define \( \text{LHS}(n) = f_{4n} \). We want to show that \( P(n) \) is true for all \( n \).

- **Basic step**:

  \[ \text{LHS}(1) = f_4 = 3 \]

  Therefore \( \text{LHS}(1) \) is divisible by 3 and \( P(1) \) is true.

- **Inductive step**: Let \( k \) be a positive integer, and let us suppose that \( P(k) \) is true. The there exist \( m \) such that \( \text{LHS}(k) = f_{4k} = 3m \). We want to show that \( P(k + 1) \) is true.

  Then

  \[ \text{LHS}(k + 1) = f_{4k+4} \]
  \[ = f_{4k+3} + f_{4k+2} \]
  \[ = 2f_{4k+2} + f_{4k+1} \]
  \[ = 2(f_{4k+1} + f_{4k}) + f_{4k+1} \]
  \[ = 3f_{4k+1} + 2f_{4k} \]
  \[ = 3f_{4k+1} + 6m \]
  \[ = 3(f_{4k+1} + 2m) \]

  Therefore \( \text{LHS}(k + 1) \) is divisible by 3, which validates that \( P(k + 1) \) is true.

The principle of proof by mathematical induction allows us to conclude that \( P(n) \) is true for all \( n \).
Others

Exercise a

Show that \(21/(4^{n+1} + 5^{2n-1})\) for all \(n > 0\).

Let \(P(n)\) be the proposition: \((4^{n+1} + 5^{2n-1})\) is divisible by 21. We define \(A(n) = 4^{n+1} + 5^{2n-1}\).

We want to show that \(P(n)\) is true for all \(n\).

- **Basis step:**

\[
A(1) = 4^2 + 5 = 16 + 5 = 21
\]

Therefore \(A(1)\) is divisible by 21 and \(P(1)\) is true.

\[
A(2) = 4^3 + 5^3 = 64 + 125 = 189 = 9 \times 21
\]

Therefore \(A(2)\) is divisible by 21 and \(P(2)\) is true.

- **Inductive step:** Let \(k\) be a positive integer, and let us suppose that \(P(k)\) is true. Then there exist \(m\) such that \(A(k) = 21m\), namely \(4^{k+1} + 5^{2k-1} = 21m\). We want to show that \(P(k+1)\) is true.

Then

\[
A(k + 1) = 4^{k+2} + 5^{2k+1} = 4 \times 4^{k+1} + 25 \times 5^{2k-1} = 4 \times (21m - 5^{2k-1}) + 25 \times 5^{2k-1} = 21 \times (4m) + 21 \times 5^{2k-1} = 21 \times (4m + 5^{2k-1})
\]

Therefore \(A(k + 1)\) is divisible by 21, which validates that \(P(k + 1)\) is true.

The principle of proof by mathematical induction allows us to conclude that \(P(n)\) is true for all \(n\).

Exercise b

Show that any postage value of \(n\) cents can be composed with a combination of 4-cent and 7-cent stamps only, when \(n\) is greater or equal to 18.

Let \(P(n)\) be the proposition: \(n\) cents can be composed with a combination of 4-cent and 7-cent stamps only.

We want to show that \(P(n)\) is true for all \(n \geq 18\).

We note first that \(P(n)\) can be rewritten as: There exits a pair of integers \((a, b)\) such that \(n = 4a + 7b\), with \(a \geq 0\) and \(b \geq 0\).

We use a proof by induction:
• **Basis step:**
  Let \( n = 18 \); we note that \( 18 = 4 + 2 \times 7 \); therefore \( P(18) \) is true
  Let \( n = 19 \); we note that \( 19 = 3 \times 4 + 7 \); therefore \( P(19) \) is true

• **Inductive step:** Let \( k \) be a positive integer; we want to show that \( P(k) \rightarrow P(k+1) \) for all \( k \geq 18 \).
  To prove this implication, we suppose that \( P(k) \) is true. Then there exist \((a, b) \in \mathbb{Z}^2\) such that \( k = 4a + 7b \), with \( a \geq 0 \) and \( b \geq 0 \).
  We want to find a similar decomposition of \( k+1 \), namely we would like to write \( k+1 = 4c + 7d \), with \( c \geq 0 \) and \( d \geq 0 \). Since \( k = 4a + 7b \), we have,
  \[
  k + 1 = 4a + 7b + 1
  \]
  We note that \( 1 = 8 - 7 = 2 \times 4 - 7 \). Therefore,
  \[
  k = 4a + 7b + 2 \times 4 - 7 = 4(a + 2) + 7(b - 1)
  \]
  Since \( a \geq 0 \), \( a + 2 \geq 0 \). However, \( b - 1 \geq 0 \) if and only if \( b \geq 1 \). We therefore distinguish two cases:

  \( b \geq 1 \).
  Let us define \( c = a + 2 \) and \( d = b - 1 \). Both \( c \) and \( d \) are positive (or 0), and \( k+1 = 4c + 7d \). Therefore \( P(k+1) \) is true.

  \( b = 0 \) Then
  \[
  k = 4a + 1
  \]
  We cannot use anymore \( 1 = 8 - 7 \), as this would introduce a 7 with a negative coefficient.
  We note however that \( 1 = 21 - 20 = 3 \times 7 - 5 \times 4 \). Therefore,
  \[
  k = 4a + 3 \times 7 - 5 \times 4 = 4(a - 5) + 3 \times 7
  \]
  Let \( c = a - 5 \) and \( d = 3 \). Obviously, \( d \geq 0 \). We note that since \( k \geq 18 \), and \( k \) is in the form \( 4a \), the smallest possible value for \( a \) is 5... therefore \( c \geq 0 \). We have therefore found two positive (or 0) integers \((c, d)\) such that \( k+1 = 4c + 7d \). Therefore \( P(k+1) \) is true..

  Therefore, in all cases, \( P(k+1) \) is true.

The principle of proof by mathematical induction allows us to conclude that \( P(n) \) is true for all \( n \). Note that the proof by induction shows us that a solution exists, but does not show us how to get that solution! This is a case of a non-constructive proof.