Midterm1 (sample1): Solutions

ECS 20 (Spring 2016)

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Part I

Exercise 1

Let us build the Truth table for \( A = (p \land q) \lor (p \land \neg q) \lor (\neg p \land q) \lor (\neg p \land \neg q) \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( p \land q )</th>
<th>( p \land \neg q )</th>
<th>( \neg p \land q )</th>
<th>( \neg p \land \neg q )</th>
<th>( (p \land q) \lor (p \land \neg q) )</th>
<th>( (p \land q) \lor (p \land \neg q) \lor (\neg p \land q) \lor (\neg p \land \neg q) )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
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</tbody>
</table>

From Column 11, we can conclude that \( A \) is a tautology.

Another approach is to use logical equivalences:

\[
(p \land q) \lor (p \land \neg q) \lor (\neg p \land q) \lor (\neg p \land \neg q) \iff (p \land (q \lor \neg q)) \lor (\neg p \land (q \lor \neg q))
\]

\[
\iff (p \land T) \lor (\neg p \land T)
\]

\[
\iff p \lor \neg p
\]

\[
\iff T
\]

Exercise 2

We can use a truth table, but it is much easier to use logical equivalences:

Note that : \( (\neg p) \lor (\neg q) \lor (\neg r) \iff \neg (p \land q \land r) \).

Therefore:

\( (p \land q \land r) \lor (\neg p) \lor (\neg q) \lor (\neg r) \iff (p \land q \land r) \lor \neg (p \land q \land r) \iff T \)

hence the proposition is a tautology.

Exercise 3

Let us build the truth table for \( B = \neg (p \rightarrow \neg q) \rightarrow \neg (p \leftrightarrow \neg q) \):

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg q )</th>
<th>( (p \rightarrow \neg q) )</th>
<th>( \neg (p \rightarrow \neg q) )</th>
<th>( p \leftrightarrow \neg q )</th>
<th>( \neg (p \leftrightarrow \neg q) )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
From Column 8, we can conclude that $B$ is a tautology.

**Part II**

**Exercise 1**

Prove or disprove that if $n$ is an odd integer, the $n^2 + 4$ is a prime number.

This proposition is probably not true, in which case we only need one counter-example to invalidate it:

Note that if $n = 11$, $n^2 + 4 = 125 = 5 \times 5 \times 5$, which is not prime. The proposition is false.

**Exercise 2**

Show that if $n$ is an integer such that $n^2 + 4n + 3$ is odd, then $n$ is even.

Let $p$ be the proposition: "$n^2 + 4n + 3$ is odd", and let $q$ be the proposition "$n$ is even".

We want to show that $p \rightarrow q$ is true. It is however very difficult to start from $p$, therefore we will use an indirect proof, i.e. we prove that the contrapositive $\neg q \rightarrow \neg p$ is true.

Therefore, we want to prove: if $n$ is odd, then $n^2 + 4n + 3$ is even.

Let $n$ be an odd integer. There exists an integer $k$ such that $n = 2k + 1$. Then:

\[
\begin{align*}
 n^2 + 4n + 3 &= (2k + 1)^2 + 4 \times (2k + 1) + 3 \\
 &= 4k^2 + 4k + 1 + 8k + 4 + 3 \\
 &= 4k^2 + 12k + 8 \\
 &= 2 \times (2k^2 + 6k + 4)
\end{align*}
\]

Therefore $n^2 + 4n + 3$ is a multiple of 2, i.e. it is even.

**Exercise 3**

Prove or disprove that $\forall n > 1$, there are no 3 integers $x$, $y$ and $z$ such that $x^n + y^n = z^n$.

The proposition is probably false, in which case we only need to find one counter-example.

Note that for $n = 2$, $3^2 + 4^2 = 5^2$, i.e. for $n = 2$, we found 3 integers $x, y$ and $z$ such that $x^2 + y^2 = z^2$.

The proposition is false.

(we could also have chosen $x = y = z = 0$).