Midterm: Solutions

ECS20 (Fall 2017)

Part I: Logic

1) Let p, q, and r be three propositions. Using truth tables or logical equivalences, indicate which (if any) of the propositions below are tautologies, contradictions, or neither.

a) \( A = (p \land q) \lor r \lor (\neg q \land \neg r) \lor (\neg p \land \neg r) \)

Let us do it first using a truth table

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>p \land q</th>
<th>\neg q \land \neg r</th>
<th>\neg p \land \neg r</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>F</td>
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<td>T</td>
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<td>F</td>
<td>F</td>
<td>F</td>
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<td>T</td>
</tr>
</tbody>
</table>

The proposition \( A \) is a tautology.

Let us do it now using logical equivalences:

\[
A \iff (p \land q) \lor r \lor (\neg q \land \neg r) \lor (\neg p \land \neg r) \\
\iff (p \land q) \lor r \lor [(\neg q \lor \neg p) \land \neg r] \\
\iff (p \land q) \lor r \lor [(\neg (p \land q) \land \neg r] \\
\iff (p \land q) \lor [(r \lor (\neg (p \land q)) \land (r \lor \neg r)] \\
\iff (p \land q) \lor [(r \lor (\neg (p \land q)) \land T] \\
\iff (p \land q) \lor (r \lor (\neg (p \land q)) \\
\iff T \lor r \\
\iff T
\]

The proposition \( A \) is a tautology.

b) \([p \lor (q \rightarrow r)] \rightarrow (p \lor q \lor r)\)

Let us define \( A = p \lor (q \rightarrow r) \) and \( B = p \lor q \lor r \). Let us build the truth table:
The proposition $A \rightarrow B$ is neither a tautology nor a contradiction.

2) Smullyan’s island
A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, “If Sally is a knight, I am a knave.” John says, “Alex and Sally are of the same type.” Sally claims, “I like chocolate.” Note that “to be of the same type” means that they are both knights or they are both knaves. Does Sally really like chocolate? Justify your answer.

Let us build the table for the possible options for Alex, John, and Sally. We then check the validity of the two statements, and finally check the consistency of the truth values for those statements with the nature of Alex and John.

<table>
<thead>
<tr>
<th>Line</th>
<th>Alex</th>
<th>John</th>
<th>Sally</th>
<th>Alex says</th>
<th>John says</th>
<th>Compatibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Knight</td>
<td>Knight</td>
<td>Knight</td>
<td>F</td>
<td>T</td>
<td>No: Alex would be a Knight who lies</td>
</tr>
<tr>
<td>2</td>
<td>Knight</td>
<td>Knight</td>
<td>Knave</td>
<td>T</td>
<td>F</td>
<td>No: John would be a Knight who lies</td>
</tr>
<tr>
<td>3</td>
<td>Knight</td>
<td>Knave</td>
<td>Knight</td>
<td>F</td>
<td>T</td>
<td>No, John would be a Knave who tells the truth</td>
</tr>
<tr>
<td>4</td>
<td>Knight</td>
<td>Knave</td>
<td>Knave</td>
<td>T</td>
<td>F</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>Knave</td>
<td>Knight</td>
<td>Knight</td>
<td>T</td>
<td>F</td>
<td>No, John would be a Knight who lies</td>
</tr>
<tr>
<td>6</td>
<td>Knave</td>
<td>Knight</td>
<td>Knave</td>
<td>T</td>
<td>T</td>
<td>No, Alex would be a Knave who tells the truth</td>
</tr>
<tr>
<td>7</td>
<td>Knave</td>
<td>Knave</td>
<td>Knight</td>
<td>T</td>
<td>F</td>
<td>No, Alex would be a Knave who tells the truth</td>
</tr>
<tr>
<td>8</td>
<td>Knave</td>
<td>Knave</td>
<td>Knave</td>
<td>T</td>
<td>T</td>
<td>No, Alex would be a Knave who tells the truth</td>
</tr>
</tbody>
</table>

Therefore Alex is a Knight and John and Sally are Knaves. Since Sally is a Knave, she lies when she says that she likes chocolate... therefore she does not like chocolate!

Part II: proofs

Exercise 1
1) Let $n$ and $m$ be two integers with $n = 1,234,605$ and $m = 2,469,210$. Without using a calculator, show that at least one of those two numbers is not a perfect square.
We use a proof by contradiction. Let us suppose that both $n$ and $m$ are perfect squares. Then there exists two integer numbers $k$ and $l$ such that $n = k^2$ and $m = k^2$. We notice now that $m = 2n$. Therefore,

$$k^2 = 2l^2$$

Taking the square root, we get:

$$k = \pm \sqrt{2}l$$

Since $m \neq 0$, $l \neq 0$. We then have:

$$\sqrt{2} = \pm \frac{k}{l}$$

with $k$ and $l$ integer, and $l \neq 0$. But this is in contradiction with $\sqrt{2}$ being irrational! Therefore the statement is true.

**Exercise 2**

a) Let $n$ be an integer. Show that $n(n+1)$ is even.

We use a proof by cases:

i) $n$ is even

There exists an integer $k$ such that $n = 2k$. Then $n(n+1) = 2k(2k+1) = 2[k(k+1)]$. Since $k(k+1)$ is an integer, $n(n+1)$ is even.

ii) $n$ is odd

There exists an integer $k$ such that $n = 2k + 1$. Then $n(n+1) = (2k+1)(2k+2) = 2[(k+1)(2k+1)]$. Since $(2k+1)(k+1)$ is an integer, $n(n+1)$ is even.

In all cases, the property is true.

b) Let $n$ and $p$ be two integers. Show that either $np$ is even, or $n^2 - p^2$ is a multiple of 8.

Again, we use a proof by cases. Let $q1 : np$ is even and $q2 : n^2 - p^2$ is a multiple of 8.

i) $n$ is even and $p$ is even.

There exists two integer numbers $k$ and $l$ such that $n = 2k$ and $p = 2l$. The $np = 4kl$. Since $2kl$ is an integer, $np$ is even: $q1$ is true.

ii) $n$ is even and $p$ is odd.

There exists two integer numbers $k$ and $l$ such that $n = 2k$ and $p = 2l + 1$. The $np = 2k(2l+1)$. Since $k(2l+1)$ is an integer, $np$ is even: $q1$ is true.

iii) $n$ is odd and $p$ is even.

There exists two integer numbers $k$ and $l$ such that $n = 2k + 1$ and $p = 2l$. The $np = 2l(2k+1)$. Since $l(2k+1)$ is an integer, $np$ is even: $q1$ is true.

iv) $n$ is odd and $p$ is odd.

There exists two integer numbers $k$ and $l$ such that $n = 2k + 1$ and $p = 2l$. The $np = 2l(2k+1)$. Since $l(2k+1)$ is an integer, $np$ is even: $q1$ is true.
There exists two integer numbers $k$ and $l$ such that $n = 2k + 1$ and $p = 2l + 1$. Note that $np$ is odd. Let us compute then $n^2 - p^2$:

\[
\begin{align*}
    n^2 - p^2 &= (2k + 1)^2 - (2l + 1)^2 \\
              &= 4k^2 + 4k + 1 - 4l^2 - 4l - 1 \\
              &= 4(k^2 - l^2 + (k - l)) \\
              &= 4(k - l)(k + l + 1)
\end{align*}
\]

Let us define $m = k + l$. The $k - l = m - 2l$. Then

\[
\begin{align*}
    n^2 - p^2 &= 4(m - 2l)(m + 1) \\
              &= 4(m(m + 1) - 8l(m + 1))
\end{align*}
\]

From 2.a) we know that $m(m + 1)$ is an even number. Let us write $m(m + 1) = 2u$, where $u$ is an integer. Then $n^2 - p^2 = 8(u - l(m + 1)$ i.e. $n^2 - p^2$ is a multiple of 8; $q_2$ is true.

In all cases, we have shown that either $q_1$ is true, or $q_2$ is true.... the property is true.

**Exercise 3**

Let $x$ be a real number. Show that if $x^3 + x^2 - 2x < 0$, then $x < 1$.

Let $x$ be a real number. Let us define:

- $p : x^3 + x^2 - 2x < 0$
- $q : x < 1$.

We want to show $p \rightarrow q$. We use an indirect proof.

Assumption: $\neg q$ is true, i.e. $x \geq 1$

We note that $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x - 1)(x + 2)$

Now we note that when $x \geq 1$, $x > 0$, $x - 1 \geq 0$ and $x + 2 > 0$; then $x^3 + x^2 - 2x \geq 0$. Therefore, $\neg p$ is true.

We have shown that $\neg q \rightarrow \neg p$ is true, therefore $p \rightarrow q$ is true.

**Extra credit**

You encounter a problem on an exam that is phrased as a multiple-choice question and you are told there is only one valid choice. Unfortunately, the question has been omitted! Here are the choices:

- a) Claim A
- b) Claim A or claim B
- c) Claim B or claim C

Clearly, the questions included 3 claims that may, or may not have been true; however, as the question is missing, you do not know what those claims are! Can you still find out what the correct choice is? Justify your answer.
The easiest way to solve this problem is to notice that only claim C can be true: if claim A was true, both answers a and b would be possible, and if claim B was true, both answers b and c would be possible. Therefore C is true, A and B are false, and the answer to the question is c).
Let us derive a proof now that is based on a truth table. Let us consider the three claims, A, B, and C, that can be True or False, and let us look at the truth values for the answers a, b, and c:
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
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As only one of a), b), and c) can be true, the only possibility is line 7; therefore claims A and B are false, claim C is true, and the answer of c).