Midterm Review
Solutions

ECS 20

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Exercise 1

Build a truth table for the proposition \((p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)\)

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<td>(p \leftrightarrow \neg q)</td>
<td>(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)</td>
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Column 6 shows that \((p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)\) is a tautology.

Exercise 2

We design different proofs of the fact that the square of an even number is an even number. Let \(p\) be the proposition "'\(n\) is an even number'" and let \(q\) be the proposition "'\(n^2\) is an even number, where \(n\) is an integer.'

(i) **Direct proof:** \(p \rightarrow q\). To prove that an implication of the form \(p \rightarrow q\) is true, it is sufficient to prove that if \(p\) is true, then \(q\) is true. Let us assume \(p\) is true, i.e. \(n\) is even. We know that there exists an (unique) integer \(k\) such that \(n = 2k\). By substitution, we get \(n^2 = 4k^2 = 2(2k^2)\). This shows that \(n^2\) is divisible by 2 and therefore even by definition. Hence \(q\) is true, and the implication is always true.

(ii) **Indirect proof:** \(\neg q \rightarrow \neg p\). In an indirect proof, we attempt to prove the contrapositive of the original implication (this is a valid proof technique, as we know that an implication and its contrapositive are equivalent). We suppose \(\neg q\) is true, i.e. \(n^2\) is odd, and we want to prove that \(\neg p\) is true, i.e. \(n\) is odd. We use our knowledge from number theory! \(n^2\) is odd means that there exists a (unique) \(k\) such that \(n^2 = 2k + 1\). Then \(n^2 - 1 = 2k\). By definition, this means that 2 divides \((n - 1)(n + 1)\). Since 2 is prime, using Euclid’s first proposition, we get that \(2/(n - 1)\) or \(2/(n + 1)\). If \(2/(n - 1)\), then there exists \(m\) such that \(n - 1 = 2m\), hence
n = 2m + 1 and n is odd, by definition. If \(2/(n+1)\), then there exists m such that \(n+1 = 2m\), hence \(n = 2m - 1\), and n is odd, by definition. In all cases, n is odd, which concludes the proof.

An even simpler proof: since 2 is prime, according to Fermat’s little theorem, \(n^2 \equiv n(\text{mod} 2)\). Hence if \(n^2 \equiv 1(\text{mod} 2)\), \(n \equiv 1(\text{mod} 2)\).

(iii) **Proof by contradiction.** Given \(p\) true, we assume that \(\neg q\) is true, and we show that we reach a contradiction. Let \(n\) be an even number, and let us assume that \(n^2\) is an odd number. There exists \(k\) such that \(n^2 = 2k + 1\). We show then (see indirect proof above) that \(n\) is odd, which contradicts the premise (i.e. we have \(p \land \neg p\), which is a contradiction). Hence the assumption \(n^2\) is odd is false, and \(n^2\) is even.

**Exercise 3**

Suppose that \(a\) is a non-zero rational number, and \(b\) is an irrational number; we want to show that the product \(ab\) is irrational. We use a proof by contradiction, i.e. we suppose that \(ab\) is rational, and we attempt to show that this leads to a contradiction. Let us write \(ab = c\), with \(c\) rational. Since \(a\) is a non-zero rational, it has a multiplicative inverse, \(a^{-1}\) that is also rational. Then \(b = ca^{-1}\). Since the product of two rational numbers is rational, this shows that \(b\) is rational, which contradicts the premise that \(b\) is irrational. Hence the hypothesis \(ab\) is rational is false, and \(ab\) is therefore irrational.

**Exercise 4**

Since there is an order relation on real numbers, given 2 real numbers, \(x\) and \(y\), there can be 3 cases, \(x > y\), \(x < y\), and \(x = y\) (this is sometimes referred to as the trichotomy law).

a) When \(x > y\), \(\max(x, y) = x\) and \(\min(x, y) = y\). In this case, \(\max(x, y) + \min(x, y) = x + y\).

b) When \(x = y\), \(\max(x, y) = \min(x, y) = x = y\). In this case, \(\max(x, y) + \min(x, y) = x + x = x + y\).

c) When \(x < y\), \(\max(x, y) = y\) and \(\min(x, y) = x\). In this case, \(\max(x, y) + \min(x, y) = y + x = x + y\), by commutative property of addition of real numbers.

The method of proof by cases allows us to conclude that \(\max(x, y) + \min(x, y) = x + y\) for all \((x, y) \in \mathbb{R}^2\).

**Exercise 5**

Let \(a = 65^{1000} - 8^{2001} + 3^{177}\), \(b = 79^{1212} - 9^{2399} + 2^{2001}\) and \(c = 24^{4493} - 5^{8192} + 7^{1777}\); we want to show that the product of two of these 3 numbers is non negative. In other words, we want to show that \(\text{ONE}\) of the elements of the set \(\{ab, ac, bc\}\) is non negative. We develop a proof by contradiction. We suppose that \(\text{ALL}\) the elements of the set \(\{ab, ac, bc\}\) are strictly negative. Let \(P\) by the product of all the elements of that set. Since there are 3 negative elements in that set, \(P\) is strictly negative. But \(P = abacbc = a^2b^2c^2\), i.e. \(P\) is the product of 3 positive numbers (three squares), hence \(P\) is positive. We have shown that \(P\) is both strictly negative and positive, i.e we have reached a contradiction. The hypothesis was wrong, and we therefore validate that the product of two of the 3 numbers a, b and c is non negative.
Exercise 6

a) $x \in A \cup B \Rightarrow x \in A \lor x \in B$. Since $A \cup B \cup C$ contains all elements either in $A$, $B$ or $C$, all
the elements of $A \cup B$ are contained in $A \cup B \cup C$. Hence, proved that $A \cup B \subset A \cup B \cup C$.

b) We know that the conjunction logic operation is both associative and commutative. Checking
membership of $(A - B) - C$:

\[
((A - B) - C) = \{ x \mid x \in ((A - B) - C) \} \\
= \{ x \mid x \in (A - B) \land \neg(x \in C) \} \\
= \{ x \mid (x \in A \land (\neg x \in B)) \land \neg(x \in C) \} \\
= \{ x \mid (\neg(x \in B) \land x \in A) \land \neg(x \in C) \} \\
= \{ x \mid \neg(x \in B) \land (x \in A \land \neg(x \in C)) \} \\
= \{ x \mid (x \in B) \land (x \in (A - C)) \}
\]

Thus, all elements of $(A - B) - C$ are contained in $(A - C)$ and not contained in $B$, which means
that all elements of $(A - B) - C$ are elements of $(A - C)$. This proves that $(A - B) - C \subset (A - C)$.

c) Let us write the definition of $(B - A) \cup (C - A)$, and use logic operations:

\[
(B - A) \cup (C - A) = \{ x \mid x \in (B - A) \lor x \in (C - A) \} \\
= \{ x \mid (x \in B \land \neg(x \in A)) \lor (x \in C \land \neg(x \in A)) \}
\]

Since $\land$ is commutative, we obtain:

\[
(B - A) \cup (C - A) = \{ x \mid (\neg(x \in A) \land x \in B) \lor (\neg(x \in A) \land x \in C) \}
\]

Since $\land$ and $\lor$ are associative, we obtain:

\[
(B - A) \cup (C - A) = \{ x \mid (\neg(x \in A)) \land (x \in B \lor x \in C) \} \\
= \{ x \mid (\neg(x \in A) \land (x \in B \cup C)) \} \\
= \{ x \mid x \in ((B \cup C) - A) \}
\]

This completes the proof that $(B - A) \cup (C - A) = (B \cup C) - A$. 