Answers to Odd-Numbered Exercises

19. a) Bit strings equal if \( x \in A \Leftrightarrow x \in B \).
   b) \( A = A' \cup \{ x \} \) if \( (A \cup \{ x \})' = A' \) and \( (A \cup \{ x \})' \neq A' \).
   c) \( x, y \in A \) if \( x \in A \) and \( y \in A \).
   d) \( (0.4, 5.6, 7.8, 9, 10) \).

20. a) The double-shaded portion is the desired set.
    b) The desired set is the entire shaded portion.
    c) The desired set is the entire shaded portion.

29. a) \( B \subseteq A \) if \( x \in B \Rightarrow x \in A \).
    b) \( A \cup B \) if \( x \in A \) or \( x \in B \).
    c) \( A \cap B \) if \( x \in A \) and \( x \in B \).
    d) Nothing, because this is always true.

39. a) \( f(x) = d \).
    b) \( f(x) = c \).
    c) \( f(x) = b \).
    d) \( f(x) = a \).

59. a) \( \{ 1, 2, 3 \} \).
    b) \( \{ x \} \).
    c) \( \{ y \} \).
    d) \( \{ z \} \).

63. \( \varnothing = \{ 0.4, 0.8 \} \).
    65. \( \varnothing = \{ 0.4, 0.8 \} \).

Section 2.3

1. a) \( f(x) \) is not defined.
    b) \( f(x) \) is not defined because there are two distinct values assigned to each \( x \).

3. a) \( f(x) \) is not a function because it is not a function.
    b) \( f(x) \) is a function.
    c) \( f(x) \) is not a function because it is not a function.
    d) \( f(x) \) is not a function because it is not a function.

5. a) \( f(x) \) is the domain of the bit strings; range the set of integers.
    b) \( f(x) \) is the domain of the bit strings; range the set of even nonnegative integers.
    c) \( f(x) \) is the domain of the bit strings; range the set of positive integers.
    d) \( f(x) \) is the domain of the bit strings; range the set of nonnegative integers.

7. a) \( f(x) \) is not onto.
    b) \( f(x) \) is onto.
    c) \( f(x) \) is not onto.
    d) \( f(x) \) is onto.

9. a) \( f(x) \) is one-to-one.
    b) \( f(x) \) is one-to-one.
    c) \( f(x) \) is one-to-one.
    d) \( f(x) \) is one-to-one.

11. a) Only the function in part (a) is one-to-one.
    b) All other functions are onto.
    c) On to.
    d) Onto.

15. a) Onto.
    b) Not onto.
    c) Onto.
    d) Not onto.

17. a) Depends on whether teachers share offices.
    b) One-to-one assuming only one teacher per bus.
    c) One-to-one assuming only one teacher per bus.

19. a) One-to-one assuming only one teacher per bus.
    b) One-to-one assuming only one teacher per bus.
    c) One-to-one assuming only one teacher per bus.
    d) One-to-one assuming only one teacher per bus.

23. a) \( f(x) \) is strictly decreasing.
    b) \( f(x) \) is not strictly decreasing.
    c) \( f(x) \) is strict increasing.
    d) \( f(x) \) is not strictly decreasing.

27. a) \( f(x) \) is a strictly decreasing function.
a < b, then f(a) ≺ f(b); if a = b, then f(a) = f(b). Thus if a < b, then f(a) ≺ f(b). Answers will vary; for example, f(x) = 0 for x < 0 and f(x) = −x for x ≥ 0.

29. The function is not one-to-one, so it is not invertible. On the restricted domain, the function is the identity function on the nonnegative real numbers, f(x) = x, so it is its own inverse.

31. a) f(S) = {0, 1, 3, 5, 8}
b) f(S) = {0, 1, 2, 4, 6, 8}
c) f(S) = {0, 1, 2, 4}
d) f(S) = {1, 2, 3, 6, 8}

33. a) Let x and y be distinct elements of A. Because y is one-to-one, f(x) and f(y) are distinct elements of B. Because f is one-to-one, f(g(x)) = f(g(y)) if and only if x = y. Therefore f(g(x)) and f(g(y)) are distinct elements of C. Hence, f ∘ g is one-to-one. b) Let y ∈ C. Because f is onto, y = f(b) for some b ∈ B. Now because y is onto, b = g(x) for some x ∈ A. Hence, f(x) = f(g(x)) = f(g(y)). It follows that f ∘ g is onto.

35. No. For example, suppose that A = {a, b}, B = {c, e}, and C = D. Let f(a) = b, f(b) = d, and f(c) = d. Then f and f ∘ g are onto, but g is not. 39. f is one-to-one because f((1 + 2) + 3) = (1 + 2 + 3) = 6, f((2 + 1) + 3) = (2 + 1 + 3) = 6, and f((2 + 1) + 3) ≠ f((1 + 2) + 3). Therefore f is onto because f((2 + 1) + 3) = (2 + 1 + 3) = 6.

41. a) 
A = B = R, S = {x | x > 0}, T = {x | x < 0}, f(x) = x. b) 
If f(S) ∩ f(T) ⊆ f(S) ∩ f(T), then because f is a one-to-one function, it follows that x ∈ f(S) for some x ∈ S. Similarly, y ∈ f(T) for some y ∈ T. Because f is onto, it follows that x = y.

45. b) (x − 1, x + 1) c) (x − 1, x + 1)

47. Let y = x + c, where c is a real number with 0 ≤ c < 1. If x = −1, then |x| = 1 < x < 0, so |x| = x = (x − 1) and this is the integer closest to x. If x = 1, then |x| = x < 1, so |x − 1| = |x + 1| and this is the integer closest to x. If x = 2, then |x| = x = (x − 1) = x = x − 1 = x − 1, which is the smaller of the two integers that surround x, and they are the same distance from x.

49. Write the real number x as x = x + c, where c is a real number with 0 ≤ c < 1. Because x = x + c, it follows that 0 ≤ c < 1. The first two inequalities, x − 2 < x and x ≤ x, follow directly. For the other two inequalities, write x = x − 1, where 0 ≤ −c < 1.

53. a) Suppose that n ≥ a. By the definition of the floor function, it follows that [x] ≤ n. This means that if n > [x], then n > a. b) Suppose that a ≥ n. By the definition of the ceiling function, it follows that [x] ≥ n. This means that if n < [x], then n < x.

55. a) Suppose that n ≥ a. By the definition of the ceiling function, it follows that [x] ≥ n. This means that if n > [x], then n > x. If n = [x], then n = [k/2] for some integer k. Thus, [k/2] = k/n. If n is odd, then n = 2k + 1 for some integer k. Thus, [k/2] = (k + 1)/2 and [k] = (n − 1)/2.

57. a) Suppose that a ≥ 0. The left-hand side is (−a − a) and the right-hand side is (−a) − a. Therefore the equation also holds because it can be obtained by substituting −a for x.

59. a) 1 b) 3 c) 126 d) 3000 e) 1091 f) 256 g) 1030

61. a) 3 b) 2 c) 1 d) 1 e) 1 f) 1

63. 65. 67.

69. f(−3) = (−3)−16(2)/3 = 1. 71. a) f(g(x)) = x = 1 → x ∈ A ∩ B → x ∈ A and x ∈ B → f(x) = 1 and f(x) = 1 = f(x) = 1 = f(x) = 1 = 1.

73. a) True, because |x| is already an integer. b) False, because |x| is already an integer.
neither $x$ nor $y$ is an integer, then $x = n + k$ and $y = m + k$ (where $n$ and $m$ are integers) and $x$ and $y$ are positive real numbers less than 1. Then $m + n + x + y < m + n + 2$, so $(x + y)$ is either $m + n + 1$ of $m + n + 2$. Therefore, the given expression is either $(m + n + 1) + (m + n + 1) = 1$ or $(m + n + 1) + (m + n + 2) = 2$, as desired. d) False: $x = 1$ and $y = 3$ is a counterexample. e) False: $x = 1$ is a counterexample. 73. If $f$ is a positive integer, then the two sides are equal. So suppose that $x = n^2 + m$, where $nx$ is the largest perfect square less than $x$, $m$ is a nonnegative integer, and $0 < x < 1$. Then both $\sqrt{nx}$ and $\sqrt{(n+1)x}$ are between $n$ and $n + 1$, so both sides equal $n$. b) If $x$ is a positive integer, then the two sides are equal. So suppose that $x = n^2 + m$, where $x$ is the smallest perfect square greater than $x$, $m$ is a nonnegative integer, and $x < 1$. Therefore, both sides of the equation are equal. 77. a) Domain is $Z$: codomain is $R$: domain of definition is the set of nonzero integers; the set of values for which $f$ is undefined is $0$; not a total function. b) Domain is $Z$: codomain is $Z$: domain of definition is $Z$: set of values for which $f$ is undefined is $Z$: not a total function. c) Domain is $Z$: codomain is $Q$: domain of definition is $Z$: set of values for which $f$ is undefined is $Z$: not a total function. d) Domain is $Z$: codomain is $Z$: domain of definition is $Z$: set of values for which $f$ is undefined is $(m, n, |m | < n)$; not a total function. 79. a) By definition, to say that $x$ is cardinality $m$ is to say that $x$ has exactly $m$ distinct elements. Therefore we can assign the first object to 1, the second to 2, and so on. This provides the one-to-one correspondence. b) By part (a), there is a bijection $f$ from $S$ to $\{1, 2, \ldots, m \}$ and a bijection $g$ from $T$ to $\{1, 2, \ldots, m \}$. Then the composition $g \circ f$ is the desired bijection from $S$ to $T$.

Section 2.4

1. a) $3$ b) $1$ c) $787$ d) $2639$ 3. a) $a_0 = 2, a_1 = 3$, $a_0 = 0, a_1 = -1, a_2 = -2, a_3 = 27, a_4 = 25$, c) $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 3, a_4 = 25$, $a_2 = 2, a_3 = 3$ 5. a) $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$ b) $1, 1, 2, 2, 3, 3, 3, 4, 4, 1, 1, 3, 3, 5, 7, 7, 9, 9$ d) $1, -1, -2, -2, 8, 88, 656, 4912, 48064, 362368, 3627776$ e) $3, 6, 12, 24, 48, 96, 192, 384, 768, 1536$ f) $2, 4, 6, 10, 16, 24, 42, 68, 110, 178$ g) $1, 1, 2, 2, 3, 3, 3, 4, 4, 4$ h) $1, 3, 3, 5, 4, 3, 5, 4, 3, 5, 7, 7, 9, 9$ 7. Each term could be twice the previous term; the $r$th term could be obtained from the previous term by adding $r$; the terms could be the positive integers that are not multiples of 3; there are infinitely many other possibilities. 9. a) $2, 2, 72, 432, 2592$ b) $2, 4, 16, 256, 65536, c) 1, 2, 5, 11, 26$ d) $1, 1, 6, 27, 204$ e) $1, 2, 0, 1, 3$ f) $a_6 = 17, 49, 143, 421$ g) $a_9 = 5$ h) $a_7 = 5$ i) $a_8 = 5$ j) $a_{n-1} = -6a_{n-2} = 5(2n^2 - 1) + 5$ 31. $-\frac{62^n - 1}{3} + 5 \cdot 2^{3n - 2} = \sum_{n=1}^{\infty} \frac{62^n - 1}{3} + 5 \cdot 2^{3n - 2}$

Section 2.5

1. a) Countably infinite, $\leq -2, -3, -4, \ldots$ b) Countably infinite, $0, 2, -2, 4, -4, \ldots$ c) Countably infinite, $99, 98, 97, \ldots$ d) Uncountable e) Finite f) Countably infinite, $0, 7, -7, 14, -14, \ldots$ 3. a) Countable: match with the string of $n$ as $b$. b) Countable: to find a correspondence, follow the path in Example 4, but omit the terms in the top three rows (as well as continuing to omit fractions not in lowest terms). c) Uncountable d) Uncountable 5. Suppose $\bar{R}$ new guests arrive at the fully occupied hotel. Move the guest in Room $n$ to Room $n + 20$ if $n = 1, 2, 3 \ldots$ then the new guests can occupy rooms $1$ to $m$.

For $n = 1, 2, 3 \ldots$, put
Then, because \( A \) onto, we merely need to show that the range of values for \( x \) real numbers from 1 to 2), \( B \) into \( \mathbb{Z} \), namely there is a one-to-one function \( f \) for some non-negative integer \( n \), in which case there is a one-to-one function from \( A \) to a subset of \( \mathbb{Z}^+ \) [the range is the first \( x \) positive integer], or there exists a one-to-one correspondence \( f \) from \( A \) to \( \mathbb{Z}^+ \), in either case we have satisfied Definition 2. Conversely, suppose that \( |A| \leq |\mathbb{Z}^+| \). By definition, this means that there is a one-to-one function from \( A \) to \( \mathbb{Z}^+ \), so \( A \) has the same cardinality as a subset of \( \mathbb{Z}^+ \) (namely the range of that function). By Exercise 16 we conclude that \( A \) is countable. 15. A sequence that \( A \) is countable. Then the elements of \( B \) can be listed as \( a_1, a_2, b_1, b_2 \). Because \( A \) is a subset of \( B \), taking the subsequence of \( (b_n) \) that contains the terms that are in \( A \) gives a listing of the elements of \( A \). Because \( A \) is uncountable, this is impossible. 17. A sequence that \( A \) is countable. Then, because \( A = (A - B) \cup (A \cap B) \), the elements of \( A \) can be listed in a sequence by alternating elements of \( A - B \) and elements of \( A \cap B \). This contradicts the uncountability of \( A \). 18. We are given bijections \( f \) from \( A \) to \( B \) and \( g \) from \( C \) to \( D \). Then the function from \( A \times C \) to \( B \times D \) that sends \( (a, c) \) to \( (f(a), g(c)) \) is a bijection. 21. By the definition of \( |A| \leq |B| \), there is a one-to-one function \( f : A \rightarrow B \). Similarly, there is a one-to-one function \( g : B \rightarrow C \). By Exercise 33 in Section 2.3, the composition \( g \circ f : A \rightarrow C \) is one-to-one. Therefore by definition \( |A| \leq |C| \). 23. Using the axiom of Choice from set theory, choose distinct elements \( a_1, a_2, a_3, \ldots \) of \( A \) at arbitrary (this is possible because \( A \) is infinite). The resulting set \( \{a_1, a_2, a_3, \ldots \} \) is the desired infinite subset of \( A \). 25. The set of all bit strings of length \( n \), for any finite alphabet is countably infinite, because we can list these strings in an alphabetical order by length. Therefore the infinite set \( X \) can be identified with an infinite subset of this countable set, which by Exercise 16 is also countably infinite. 27. Suppose that \( A_1, A_2, A_3, \ldots \) are countable sets. Because \( A_1 \) is countable, we can list its elements \( a_i \) in a sequence as \( a_1, a_2, a_3, \ldots \). The elements of the set \( A_1 \times A_2 \times A_3 \times \cdots \) can be listed by listing all terms \( a_i \) with \( i + j + \cdots + k = 1 \), then all terms \( a_j \) with \( i + \cdots + j + k = 2 \), then all terms \( a_k \) with \( i + \cdots + k = 3 \), then all terms \( a_j \) with \( i + \cdots + j = 4 \), and so on. 29. There are a finite number of bit strings of length \( n \), namely \( 2^n \). The set of all bit strings is the union of the sets of bit strings of length \( n \) for \( n = 0, 1, 2, \ldots \). Because the union of a countable number of countable sets is countable (see Exercise 27), there are a countable number of bit strings. 31. It is clear from the formula that the range of values the function takes on for a fixed value of \( m = n \). Say \( m + n = s \), is \( x = 2(x_1 - 1) + 2 + 1 \) through \( x = 2(x_1 - 1) + 2 + (s - 1) \), because \( m \) can assume the values 1, 2, 3, \ldots \( (s - 1) \) under these conditions, and the first term in the formula is a fixed positive integer when \( m + n = s \). To show that this function is one-to-one and onto, we merely need to show that the range of values for \( x = 1 \) picks up precisely where the range of values for \( x \) left off, i.e., that \( \{x(1, 1) = 1 + f(x, 1, 1) \} \). We have \( f(x - 1, 1, 1) + 1 = f(x, 1, 1) + 1 = x^2 - x + 1 = \frac{x^2 - 1}{x-1} + 1 = f(1, x) \). By the Schröder-Bernstein theorem, it suffices to fix one-to-one functions \( f(0, 1) \rightarrow (0, 1) \) and \( g : [0, 1) \rightarrow (0, 1) \). Let \( f(x) = x \) and \( g(x) = (x + 1)/3 \). Each element \( A \) of the power set of the set of positive integers \( (i.e., A \subseteq \mathbb{Z}^+) \) can be represented uniquely by the bit string \( a_{n_1}a_{n_2}a_{n_3} \ldots \) where \( a_i = 1 \) if \( i \in A \) and \( a_i = 0 \) if \( i \notin A \). A sequence there was a one-to-one correspondence \( f: \mathbb{Z}^+ \rightarrow \mathbb{N}(Z^+) \). Form a new bit string by setting \( s_i \) to be 1 minus the \( i \)th bit of \( f(x) \). Then because \( i \) differs in the \( i \)th bit from \( f(x) \), it is not in the range of \( f \), a contradiction.

For any finite alphabet there are a finite number of strings of length \( n \), whenever \( n \) is a positive integer. It follows by the result of Exercise 27 that there are only a countable number of strings from any given finite alphabet. Because the set of all computer programs in a particular language is a subset of the set of all strings of a finite alphabet, which is a countable set by the result from Exercise 16, it is itself a countable set. 39. Exercise 37 shows that there are only a countable number of computer programs. Consequently, there are only a countable number of computable functions. Because, as Exercise 38 shows, there are an uncountable number of functions, not all functions are computable.

### Section 2.6

1. a) \( 3 \times 4 \) \hspace{1cm} b) \( 1 \) \hspace{1cm} c) \( \{0, 4, 6\} \) \hspace{1cm} d) 1

2. a) \( 1 \) \hspace{1cm} b) \( 2 \) \hspace{1cm} c) \( \{0, 4, 6\} \) \hspace{1cm} d) 1

3. a) \( 1 \) \hspace{1cm} b) \( 2 \) \hspace{1cm} c) \( \{-4, 5, 9\} \) \hspace{1cm} d) \( \{-1, 5, 4, 5\} \)

7. 0 \hspace{1cm} 1 \hspace{1cm} a) \( [0, a_1] \) \hspace{1cm} b) \( \{a_1\} \) \hspace{1cm} c) \( A \) \hspace{1cm} d) \( \{a_1, a_2, a_3\} \) \hspace{1cm} e) \( A + B + C \) \hspace{1cm} f) \( A \cap B \cap C \)

11. The number of rows of \( A \) equals the number of columns of \( B \), and the number of columns of \( A \) equals the number of rows of \( B \). 13. \( A (BC) = A (B + C) = A (B + C) = A B + A C \)

15. \( A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) \hspace{1cm} 17. a) Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \). Then \( A + B = [a_{ij} + b_{ij}] \). We have \( A = B \in \mathbb{R} \).

b) Using the same notation as in part (a), we have \( B'A' = B'A' \).