CHAPTER 5

Section 5.3

1. Let \( P(n) \) be the statement that the train stops at station \( n \). Basis step: We are told that \( P(1) \) is true. Inductive step: We are told that \( P(n) \) implies \( P(n + 1) \) for each \( n \). Therefore by the principle of mathematical induction, \( P(n) \) is true for all positive integers \( n \).

3. Assume by the inductive hypothesis that \( k^2 + 1 \) is divisible by 2. Then \( (k + 1)^2 + 1 = k^2 + 2k + 1 + 1 = k^2 + 2k + 2 = (k^2 + 1) + 2k \). By the induction hypothesis, \( (k^2 + 1) \) is divisible by 2, so \( (k + 1)^2 + 1 \) is divisible by 2.

4. b) Let \( n \geq 2 \). Then \( 12 \cdot n + 1 \) is divisible by 2. By the induction hypothesis, \( 12 \cdot n \) is divisible by 2. Therefore \( 12 \cdot n + 1 \) is divisible by 2.

8. Assume by the inductive hypothesis that \( (k + 1)^3 \) is divisible by 3. Then \((k + 2)^3 = k^3 + 3k^2 + 3k + 2 = (k^3 + 1) + 3k^2 + 3k + 1 = (k^3 + 1) + 3((k + 1)^2) + 3(k + 1) + 1 \). By the induction hypothesis, \( (k^3 + 1) \) is divisible by 3, and \( 3(k + 1)^2 \) and \( 3(k + 1) \) are both divisible by 3. Therefore \((k + 2)^3 \) is divisible by 3.

11. Assume by the inductive hypothesis that \( n \cdot n! \) is divisible by \( n \). Then \( (n + 1)(n + 1)! = (n + 1)n! + n! = n! \cdot (n + 1 \cdot 1 + 1) = n! \cdot (n + 2) \). By the induction hypothesis, \( n! \) is divisible by \( n \), and \( n + 2 \) is divisible by 1. Therefore \( (n + 1)(n + 1)! \) is divisible by \( n + 1 \).

14. Assume by the inductive hypothesis that \( n \cdot n! \) is divisible by \( n \). Then \( (n + 1)(n + 1)! = (n + 1)n! + n! = n! \cdot (n + 1 \cdot 1 + 1) = n! \cdot (n + 2) \). By the induction hypothesis, \( n! \) is divisible by \( n \), and \( n + 2 \) is divisible by 1. Therefore \( (n + 1)(n + 1)! \) is divisible by \( n + 1 \).

15. Let \( P(n) \) be \( 2n^2 + 1 \) is divisible by 3. Then \( 2(n + 1)^2 + 1 = 2n^2 + 4n + 2 + 1 = 2n^2 + 1 + 4(n + 1) = (2n^2 + 1) + 4(n + 1) \). By the induction hypothesis, \( 2n^2 + 1 \) is divisible by 3, and \( 4(n + 1) \) is divisible by 3. Therefore \( 2(n + 1)^2 + 1 \) is divisible by 3.
the sum of a multiple of 2 (by the inductive hypothesis) and a multiple of 2 (by definition), hence, divisible by 2. 33. Let $P(n)$ be \( n^2 - n \) is divisible by 5. Basis step: $P(0)$ is true because \( 0^2 - 0 = 0 \) is divisible by 5. Inductive step: Assume that $P(k)$ is true, that is, \( n^2 - n \) is divisible by 5. Then \((k^2 - k) = (k^2 + 5k + 10) - 5(k^2 + 1) = (k^2 - k) + 5(k+1)\) is also divisible by 5, because both terms in this sum are divisible by 5. 35. Let $P(n)$ be the proposition that $2n - 1$ is divisible by 8. The basis case $P(1)$ is true because $8 \mid 0$. Now assume that $P(n)$ is true. Because $(2n + 1) - 1 = 2(n + 1)$, $P(n + 1)$ is true because both terms on the right-hand side are divisible by 8. This shows that $P(n)$ is true for all positive integers $n$, so $n^2 - n$ is divisible by 8 whenever $n$ is an odd positive integer. 37. Basis step: $1111 + 1221 + 1211 = 1212 + 1133$ Inductive step: Assume the inductive hypothesis, that $1111 + 1221 + 1211 \cdots + 1111 + 1221 + 1211 = 1133 + 1233$. The expression in parentheses is divisible by 133 by the inductive hypothesis, and obviously the second term is divisible by 133, so the entire quantity is divisible by 133, as desired. 39. Basis step: $A_1 \subseteq B_1$, tautologically implies that \( \bigcap_{i=1}^n A_i \subseteq \bigcap_{i=1}^n B_i \). Inductive step: Assume the inductive hypothesis that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k$, then \( \bigcup_{i=1}^k A_i \subseteq \bigcup_{i=1}^k B_i \). We want to show that if $A_k \subseteq B_k$, then $\bigcup_{i=1}^k A_i \subseteq \bigcup_{i=1}^k B_i$. Let $x$ be an arbitrary element of $\bigcup_{i=1}^k A_i$, i.e., $x \in A_k$ by the inductive hypothesis. Thus $x \in A_k \subseteq B_k$, we know from the given fact that $A_{k+1} \subseteq B_{k+1}$, that $x \in B_{k+1}$, therefore, $x \in \bigcup_{i=1}^{k+1} B_i \subseteq \bigcup_{i=1}^k B_i$. 41. Let $P(n)$ be \( \mathbb{A}_{A_1} (\bigcup_{i=1}^n A_i) \cap \bigcup_{i=1}^n B_i \). Assume that $P(k)$ is true. Then $A_1 \cup A_2 \cup \cdots \cup A_k \cap B_k$ is an integer greater than or equal to 0. 43. Basis step: $P(1)$ is trivially true. Inductive step: Assume that $P(k)$ is true. Then $(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}) \cap B_k = (A_1 \cup A_2 \cup \cdots \cup A_k) \cap B_k \cup A_{k+1}$. 45. Let $P(n)$ be the statement that a set with $n$ elements has $n(n-1)/2$ two-element subsets. By $P(2)$, the basis case is true, because a set with two elements has one subset with two elements—namely, itself—and $2/2 = 1$. Now assume that $P(k)$ is true. Let $S$ be a set with $k + 1$ elements. Choose an element $s$ in $S$ and let $T = S - \{s\}$. A two-element subset of $S$ either contains $s$ or does not. Those subsets not containing $s$ are the subsets of $T$ with two elements; by the inductive hypothesis there are $k + 1 - 2 = k - 1$ of these. There are $k$ subsets of $S$ with two elements that contain $s$, because such a subset contains $s$ and one of the $k$ elements in $T$. Hence, there are $k + 1 - 1 + 2(k - 1)/2$ two-element subsets of $S$. This completes the inductive proof.
tion says that \( m^{k+1} \equiv b^{k+1} \pmod{m} \). 63. Let \( P(n) \) be \((p_1 \rightarrow p_2) \land (p_2 \rightarrow p_3) \land \cdots \land (p_{k-1} \rightarrow p_k) \). Basis step: \( P(1) \) is true because \((p_1 \rightarrow p_2) \land (p_2 \rightarrow p_3) \land \cdots \land (p_{k-1} \rightarrow p_k) \) is a tautology. Inductive step: A stronger \( P(n) \) is true. To show \((p_1 \rightarrow p_2) \land (p_2 \rightarrow p_3) \land \cdots \land (p_{k-1} \rightarrow p_k) \Land (p_k \rightarrow p_{k+1}) \rightarrow (p_1 \rightarrow p_{k+1}) \equiv (p_1 \rightarrow p_{k+1}) \), assume the hypothesis of this conditional statement is true. Because both the hypothesis and \( P(k) \) are true, it follows that \((p_1 \land \cdots \land p_{k-1}) \rightarrow p_k \) is true. Because this is true, and because \( p_{k-1} \rightarrow p_{k+1} \) is true (it is part of the assumption) it follows by hypothetical syllogism that \((p_1 \land \cdots \land p_{k-1}) \rightarrow p_{k+1} \) is true. The weaker statement \((p_1 \rightarrow p_{k+1}) \rightarrow p_{k+1} \), follows from this. 63. We will first prove the result when \( n \) is a power of 2, that is, if \( n = 2^k \), \( k = 1, 2, \ldots \). Let \( P(n) \) be the statement \( A \geq G \), where \( A \) and \( G \) are the arithmetic and geometric means, respectively, of a set of \( n = 2^k \) positive real numbers. Basis step: \( k = 1 \) and \( n = 2 \). Note that \( \sqrt[n]{2^k} \geq 0 \). Expanding this shows that \( a_1 = 2 - \sqrt[2]{a_2} \geq 0 \), that is, \( (a_1 - 0.5)^2 \geq 0 \). Inductive step: Assume that \( P(k) \) is true, with \( n = 2^k \). We will show that \( P(k+1) \) is true. We have \( n = 2^{k+1} \). Now \( a_{1} + a_{2} + \ldots + a_{n} = (a_{1} + a_{2} + \ldots + a_{m}) + (a_{m+1} + a_{m+2} + \ldots + a_{2m}) / 2 \) and similarly \( (a_{2m+1} + a_{2m+2} + \ldots + a_{2^n}) / 2 \). To simplify the notation, let \( A(x, y, \ldots) \) and \( G(x, y, \ldots) \) denote the arithmetic and geometric means of \( x, y, \ldots \), respectively. Assume that \( a_{1} \leq a_{2} \leq \ldots \leq a_{n} \leq y, y \), and so on, then \( A(x, y, \ldots) \leq G(x, y, \ldots) \leq G(y, y, \ldots) \). Hence, \( A(a_{1}, a_{2}, \ldots, a_{n}) = A(a_{1}, a_{2}, \ldots, a_{m}, a_{m+1}, \ldots, a_{2m}) \geq G(a_{1}, a_{2}, \ldots, a_{m}, a_{m+1}, \ldots, a_{2m}) \geq G(a_{1}, a_{2}, \ldots, a_{n}) \). This finishes the proof for powers of 2. Now if \( n \) is not a power of 2, let \( m \) be the next higher power of 2, and let \( a_{1}, a_{2}, \ldots, a_{n} \) all equal \( a_{1}, a_{2}, \ldots, a_{m} \). Then we have \((a_{1} + a_{2} + \ldots + a_{m}) / m \leq A(a_{1}, a_{2}, \ldots, a_{m}) \) because \( m \) is a power of 2. Because \( A(a_{1}, a_{2}, \ldots, a_{m}) = \pi \), it follows that \( a_{1} + a_{2} + \ldots + a_{m} / m \leq \pi \). Raising both sides to the power \( 1 / k \) we get \( A(a_{1}, a_{2}, \ldots, a_{m}) \leq \pi^{1 / km} \). The statement is true for \( k \rightarrow k+1 \) if \( k \rightarrow k+1 \). Basis step: For \( n = 1 \), the left-hand side is just \( \frac{1}{2} \), which is 1. For \( n = 2 \), there are three nonempty subsets \( \{1\}, \{2\}, \text{and} \{1, 2\} \). The left-hand side is \( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{1}{2} \). Inductive step: Assume that the statement is true for \( k \). The set of the first \( k+1 \) positive integers has many nonempty subsets, but they fall into three categories: a nonempty subset of the first \( k \) positive integers together with \( k+1 \), a nonempty subset of the first \( k \) positive integers, or just \( k+1 \). By the inductive hypothesis, the sum of the first \( k \) integers is \( k(k+1) \). For the second category, we can factor out \( 1/(k+1) \) from each term of the sum and what remains is just \( k \) by the inductive hypothesis, so this part of the sum is \( k(k+1) \). Finally, the third category simply yields \( 1/(k+1) \). Hence, the entire summation is \( k(k+1) + 1/(k+1) = k+1 \). 67. Basis step: If \( A_1 \subseteq A_2 \), then \( A_1 \) satisfies the condition of being a subset of each in the collection; otherwise \( A_1 \not\subseteq A_2 \), so \( A_2 \) satisfies the condition. Inductive step: Assume the inductive hypothesis, that the conditional statement is true for \( k \), and suppose we are given \( k+1 \) sets that satisfy the given conditions. By the inductive hypothesis, there must be a set \( A_j \) for some \( j \leq k \) such that \( A_j \subseteq A_f \) for \( 1 \leq j \leq k \). If \( A_f \not\subseteq A_2 \), then we are done. Otherwise, we know that \( A_{j+1} \subseteq A_2 \), and this tells us that \( A_{j+1} \) satisfies the condition of being a subset of \( A_f \) for \( 1 \leq j+1 \leq k+1 \). 69. \( G(1) = 0, G(2) = 1, G(3) = 3, G(4) = 4 \). 71. To show that \( 2n-4 \) calls are sufficient to exchange all the gossip, select persons 1, 2, 3, and 4 to be the central committee. Every person outside the central committee calls one person on the central committee. At this point the central committee members as a group know all the scandals. They then exchange information among themselves by making the calls 1-2, 3-4, 1-2, and 2-4 in that order. At this point, every central committee member knows all the scandals. Finally, again every person outside the central committee calls one person on the central committee, at which point everyone knows all the scandals. (The total number of calls is \((n-4)+4+(n-4)+2 = 4(n-4) \). That cannot be done with fewer than \(2n-4 \) calls is much harder to prove, see Sandra M. Hedetniemi, Stephen T. Hedetniemi, and A. R. Liestman, “A survey of gossiping and broadcasting in communication networks,” Networks 18 (1988), no. 4, 319–349, for details. 73. We prove this by mathematical induction. The basis step (\( n = 2 \)) is true tautologically. For \( n = 3 \), suppose that the intervals are \( a_1, b_1, c_1, d_1, e_1, \) and \( e_2, f_2, \) without loss of generality we can assume that \( a_1 < e_1 < e_2 < f_2 \). Because \((a_1, e_1, f_2) \neq \emptyset \), we must have \( c_1 < e_1 < f_2 \) for a similar reason, \( a_1 < d_1 < e_2 \). It follows that the number halfway between \( e_1 \) and the smaller of \( a_1 \) and \( d_1 \) is common to all three intervals. Now for the inductive step, assume that whenever we have \( k \) intervals that have pairwise nonempty intersections then there is a point common to all the intervals, and suppose that a pair of given intervals \( I_1, I_2, \ldots, I_{k-1} \) that have pairwise nonempty intersections. For each \( i \) from 1 to \( k \), let \( \ell_i = I_i \cap I_{i+1} \). We claim that the collection \( I_1, \ell_2, \ldots, \ell_k \) satisfies the inductive hypothesis, that is, that \( \ell_1 \cap \ell_2 \cap \cdots \cap \ell_k \neq \emptyset \) for each choice of subscripts \( i_1 \) and \( i_2 \). This follows from the \( n = 3 \) case proved above, using the sets \( \ell_1, \ell_2, \ell_3 \). We can now invoke the inductive hypothesis to conclude that there is a number common to all of the sets \( I \) for \( i = 1, 2, \ldots, k \). This perforce is in the intersection of all the sets \( I \) for \( i = 1, 2, \ldots, k \). 75. Pair up the people. Have the people stand at mutually distinct small distances from their partners but far away from everyone else. Then each person throws a pie at his or her partner, so everyone gets hit. 77. Answers to Odd-Numbered Exercises 531
that are $2 = 2 \times 2$ cubes each with a $1 \times 1 \times 1$ cube removed. The basis step, $P(1)$, holds because one tile coincides with the solid to be tiled. Now assume that $P(k)$ holds. Now consider a $2^{k+1} \times 2^{k+1} \times 2^{k+1}$ cube with a $1 \times 1 \times 1$ cube removed. Split this object into eight pieces using planes parallel to its faces and running through its center. The missing $1 \times 1 \times 1$ piece occurs in one of these eight pieces. Now position one tile with its center at the center of the large object so that the missing $1 \times 1 \times 1$ cube lies in the octant in which the large object is missing a $1 \times 1 \times 1$ cube. This creates eight $2^k \times 2^k \times 2^k$ cubes, each missing a $1 \times 1 \times 1$ cube. By the inductive hypothesis we can fill each of these eight objects with tiles. Putting these tilings together produces the desired tiling.

### Section 5.2

#### 83. Let $Q(n)$ be $P(n) + b = 1$. The statement that $P(n)$ is true for $n = b, b + 1, b + 2, \ldots$ is the same as the statement that $Q(n)$ is true for all positive integers $n$. We are given that $Q(b)$ is true (i.e., that $Q(1)$ is true), and that $Q(k) \rightarrow Q(k + 1)$ for all $k \geq b$ (i.e., that $Q(m) \rightarrow Q(m + 1)$ for all positive integers $m$). Therefore, by the principle of mathematical induction, $Q(n)$ is true for all positive integers $n$.

**Basis step:** $Q(1)$ is true because we can form 9 cents of postage with one 3-cent stamp and one 5-cent stamp. $P(1)$ is true, because we can form 9 cents of postage with one 3-cent stamp and one 5-cent stamp.

**Inductive step:** Assume that $P(k)$ is true. Then $Q(k)$ is true; we can form $k$ cents of postage using just 3-cent and 5-cent stamps. Because $k > 1$, then the inductive hypothesis tells us that we can run $k - 1$ miles, so we can run $(k - 1) + 2 = k + 1$ miles. 3. a) $P(8)$ is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. $P(9)$ is true, because we can form 9 cents of postage with one 3-cent stamp and one 5-cent stamp. b) The statement that using just 3-cent and 5-cent stamps we can form $j$ cents postage for all $j$ with $8 \leq j \leq k$, where we assume that $k \geq 10$; 3. a) Assuming the inductive hypothesis, we can form $k - 1$ cents postage using just 3-cent and 5-cent stamps. b) Because $k \geq 10$, we know that $P(k - 2)$ is true, that is, that we can form $k - 2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k - 1$ cents of postage. c) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer $n$ greater than or equal to 8. 5. a) 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, and all values greater than or equal to 30. b) Let $P(n)$ be the statement that we can form $n$ cents of postage using just 4-cent and 11-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 30$. For the basis step, $n = 30 \equiv 11 + 11 + 4 + 4 + 4$. A saume that we can form $k$ cents of postage (the inductive hypothesis); we will show how to form $k + 1$ cents of postage. If the $k$ cents included an 11-cent stamp, then replace it by three 4-cent stamps. Otherwise, 4 cents was formed from just 4-cent stamps. Because $k = 30$, there must be at least eight 4-cent stamps involved. Replace eight 4-cent stamps by three 11-cent stamps, and we have formed $k + 1$ cents in postage. c) $P(n)$ is the same as in part (b). To prove that $P(n)$ is true for all $n \geq 30$, we check for the basis step that 30 = 11 + 11 + 4 + 4, 31 = 11 + 4 + 4 + 4 + 4, 32 = 4 + 4 + 4 + 4 + 4 + 4, and 33 = 11 + 11 + 11 + 11. For the inductive step, assume the inductive hypothesis, that $P(j)$ is true for all $j$ with $30 \leq j \leq k$, where $k$ is an arbitrary integer greater than or equal to 33. We want to show that $P(k + 1)$ is true. Because $k - 3 > 0$, we know that $P(k - 3)$ is true, that is, that we can form $k - 3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k + 1$ cents of postage. In this proof, our inductive hypothesis was that $P(30)$ was true for all values of $n$ between 30 and $k$ inclusive, rather than just that $P(30)$ was true. 7. We can form all amounts except $31$ and $33$. Let $P(n)$ be the statement that we can form $n$ dollars using just 2-dollar and 5-dollar bills. We want to prove that $P(n)$ is true for all $n \geq 5$. (It is clear that $1$ and $3$ cannot be formed and that $2$ and $4$ can be formed.) For the basis step, note that $5 = 2 + 2 + 1$. Assume that $P(n)$ is true for all $n \geq 5$. We want to prove that $P(n + 1)$ is true. Assume that $P(n - 3)$ is true, that is, that we can form $n - 3$ cents of postage. Put one more 5-cent stamp on the envelope, and we have formed $n + 1$ dollars. By the same reasoning as before, it is even, so $b = 2x$ for some positive integer $x$. Then $2^n = (k + 1)^2$, so $(k + 1)^2$ is even, and hence, $k + 1$ is even. So write $k + 1 = 2l$ for some positive integer $l$, whence $2^n = 4l^2$ and $k^2 = 2^n - 1$. By the same reasoning as before, $h$ is even, so $h = 2x$ for some positive integer $x$. Then $\sqrt{2^n} = \sqrt{(k + 1)/b} = (2x)/(2x) = 1/1$. But $k + 1$, so this contradicts the inductive hypothesis, and our proof of the inductive step is complete. 11. Basis step: There are four base cases. If $n = 1$, then clearly the second player wins. If there are two, three, or four matches ($n = 2, 3, 4$), then the first player can win by removing all but one match. Inductive step: Assume the strong inductive hypothesis, that in games with $n$ or fewer matches, the first player can win if $n = 2, 3, 4$. We will show how to form $k + 1$ matches, where $k \geq 4$. If $k + 1 = 0$ (mod 4), then the first player can remove three matches, leaving a $k - 2$ matches for the other player. Because $k - 2 = 1$ (mod 4), by the inductive hypothesis, this is a game that the second player wins.
at that point (who is the first player in our game) can win. Similarly, if \( k + 1 \equiv 2 \pmod{4} \), then the first player can remove one match, and if \( k + 1 \equiv 3 \pmod{4} \), then the first player can remove two matches. Finally, if \( k + 1 \equiv 1 \pmod{4} \), then the first player must leave \( k - 1 \), or \( k - 2 \) matches for the other player. Because \( k \equiv 0 \pmod{4} \), \( k - 1 \equiv 3 \pmod{4} \), and \( k - 2 \equiv 2 \pmod{4} \), by the inductive hypothesis, this is a game that the first player at that point (who is the second player in our game) can win. Let \( P(n) \) be the statement that exactly \( n \) moves are required to assemble a puzzle with \( n \) pieces. Now \( P(1) \) is trivially true. A ssume that \( P(j) \) is true for all \( j \leq k \), and consider a puzzle with \( k + 1 \) pieces. The final move must be the joining of two blocks, of size \( j \) and \( s + 1 - j \) for some integer \( j \) with \( 1 \leq j \leq k \). By the inductive hypothesis, it required \( j - 1 \) moves to construct the one block, and \( k - 1 + j - 1 = k - j \) moves to construct the other. Therefore, \( 1 + (j - 1) + (k - j) \) moves are required in all, so \( P(k + 1) \) is true. Let \( \lambda \) be the number of rows and \( \mu \) columns. We claim that the first player can win the game by making the first move to leave just the top row and leftmost column. Let \( P(n) \) be the statement that if a player has presented his opponent with a Chomp configuration consisting of just \( \alpha \) cookies in the top row and \( \alpha \) cookies in the leftmost column, then he can win the game. We will prove \( \forall \alpha \ P(\alpha) \) by strong induction. We know that \( P(1) \) is true, because the opponent is forced to take the poisoned cookie at his first turn. Fix \( k \geq 1 \) and assume that \( P(j) \) is true for all \( j \leq k \). We claim that \( P(k + 1) \) is true. It is the opponent’s turn to move. If she picks the poisoned cookie, then the game is over and she loses. Otherwise, assume she picks the cookie in the top row in column \( j \), or the cookie in the top row in column \( j \), respectively. This leaves the position \( P(j) \) with \( k + 1 \) moves. By the inductive hypothesis, this is a game that the first player can win. Let \( P(n) \) be the statement that if a simple polygon with \( n \) sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon. We will prove \( \forall n \geq 4 \ P(n) \). The statement is clearly true for \( n = 4 \), because there is only one diagonal. Leaving two triangles with the desired property. Fix \( k \geq 4 \) and assume that \( P(j) \) is true for all \( j \leq k \). Consider a polygon with \( k + 1 \) sides, and some triangulation of it. Pick one of the diagonals in this triangulation. First suppose that this diagonal divides the polygon into one triangle and one polygon with \( k + 1 \) sides. Then the triangle has two sides that border the exterior. Furthermore, the \( k \)-gon has, by the inductive hypothesis, two triangles that have two sides that border the exterior of that \( k \)-gon, and only one of these triangles can fail to be a triangle that has two sides that border the exterior; it may happen in each case that the triangle we are guaranteed in fact borders the diagonal (which is part of the boundary of that polygon). This leaves us with no triangles guaranteed to touch the boundary of the original polygon. b) We proved the stronger statement \( \forall n \geq 4 \ P(n) \) in Exercise 17.

23. a) In the left figure \( \triangle abc \) is smallest, but \( \square \) is not an interior diagonal. b) In the right figure \( \square \) is not an interior diagonal. c) In the right figure \( \triangle \) is not an interior diagonal.

24. a) When we try to prove the inductive step and find a triangle in each subpolygon with at least two sides bordering the exterior, it may happen in each case that the triangle we are guaranteed in fact borders the diagonal (which is part of the boundary of that polygon). This leaves us with no triangles guaranteed to touch the boundary of the original polygon. b) We proved the stronger statement \( \forall n \geq 4 \ P(n) \) in Exercise 17.
The inductive step then forces $P(n, k)$ to be true, a contradiction.

The last case is similar. If $x_{i}$ is a counterexample with $n$ as small as possible, we cannot have $n = 1$, because we are given that $P(1, k)$ is true for all $k$. Therefore, $n > 1$. By our choice of counterexample, we know that $P(n, k)$ is true for all $k$. Therefore, $k > 1$. By our choice of counterexample, we know that $P(n, k - 1)$ is true. But the inductive step then forces $P(n, k)$ to be true, a contradiction. 33. Let $P(n)$ be the statement that if $x_{1}, x_{2}, \ldots, x_{n}$ are distinct real numbers, then $n - 1$ multiplications are used to find the product of these numbers no matter how parentheses are inserted in the product. We will prove that $P(n)$ is true using strong induction. The base case $P(1)$ is true because $1 - 1 = 0$ multiplications are required to find the product of $x_{1}$, a product with only one factor. Suppose that $P(k)$ is true for $1 \leq k \leq n$. The last multiplication used to find the product of the first $k$ of these numbers for some $k$ and the product of the last $n - 1 - k$ of them. By the inductive hypothesis, $k - 1$ multiplications are used to find the product of a of the numbers, no matter how parentheses were inserted in the product of these numbers, and $k - k$ multiplications are used to find the product of the other $n - 1 - k$ of them, no matter how parentheses were inserted in the product of these numbers. Because one more multiplication is required to find the product of all $n + 1$ numbers, the total number of multiplications used equals $(k - 1) + (n - k + 1) = n$. Hence, $P(n)$ is true for $n$. 37. Assume that $a = dq + r = dq' + r'$ with $0 \leq r < d$ and $0 \leq r' < d$. Then $d(q - q') = r - r'$. It follows that $d$ divides $r - r'$. Because $d = r' < r < d$, we have $r' = r = 0$. Hence, $r = 0$. It follows that $q = q'$. 38. This is a paradox caused by self-reference. The answer is clearly "no." There are a finite number of English words, so only a finite number of strings of 15 words or fewer; therefore, only a finite number of positive integers can be described, not all of them. 41. Suppose the well-ordering property were false. Let $S$ be a nonempty set of nonnegative integers that has no least element. Let $P(n)$ be the statement "$n \notin S$ for $n = 0, 1, \ldots, n". P(0)$ is true because if $0 \notin S$ then $S$ has a least element, namely, 0. Now suppose that $P(n)$ is true. Then $0 \notin S$ and $n \notin S$. Clearly, $n + 1$ cannot be in $S$, for if it were, it would be its least element. Thus $P(n + 1)$ is true. So by the principle of mathematical induction, $n \notin S$ for all nonnegative integers $n$. Thus, $S = \emptyset$, a contradiction.

43. Strong induction implies the principle of mathematical induction, for if one has shown that $P(k) \Rightarrow P(k + 1)$ is true, then one has also shown that $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots$ is true. By Exercise 41, the principle of mathematical induction implies the well-ordering principle. Therefore by assuming strong induction as an axiom, we can prove the well-ordering principle.

Section 5.3

1. a) $f(1) = 3, f(2) = 5, f(3) = 7, f(4) = 9, f(5) = 11, f(6) = 13$.
   b) $f(1) = 1, f(2) = 2, f(3) = 4, f(4) = 7, f(5) = 11, f(6) = 16, f(7) = 22, f(8) = 30, f(9) = 39$.
   c) $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 5, f(6) = 8, f(7) = 13, f(8) = 21, f(9) = 34$.
   d) $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 4, f(5) = 5, f(6) = 6, f(7) = 7, f(8) = 8, f(9) = 9$.

2. a) $P(n, k)$ is true for all $n$ and $k$.
   b) $P(n, k)$ is true for all $n$ and $k$.
   c) $P(n, k)$ is true for all $n$ and $k$.
   d) $P(n, k)$ is true for all $n$ and $k$.

3. a) $P(n, k)$ is true for all $n$ and $k$.
   b) $P(n, k)$ is true for all $n$ and $k$.
are many possible correct answers. We will supply relatively simple ones.  

a) $a_{n+1} = a_n + 2$ for $n \geq 1$ and $a_1 = 6$  

b) $a_{n+1} = a_n + 2$ for $n \geq 2$ and $a_2 = 6$. For $n \geq 1$ and $a_1 = 6$  

c) $a_{n+1} = a_n - 3$ for $n \geq 1$ and $a_1 = 9$  

d) $a_{n+1} = a_n + 3$ for $n \geq 1$ and $a_1 = 0$  

9. $P(a) = 0$, $P(a + 1) = P(a) + n$  

13. Let $P(n)$ be "$f_1 + f_2 + \cdots + f_{n-1} = f_n$". Basis step: $P(1)$ is true because $f_1 = 1 = f_2$. Inductive step: A sum that $f_1 + f_2 + \cdots + f_{n-1} + f_n = f_{n+1}$ is the number of divisions used by the Euclidean algorithm to find $gcd(f_{n+1}, f_n)$.  

15. Basis step: $f_0 + f_1 + f_2 = 0 + 1 + 1 = 1 = f_2$. Inductive step: A sum that $f_0 + f_1 + f_2 + \cdots + f_{n-1} + f_{n+1} = f_{n+2}$ is the number of divisions by $f_{n+1}$ that the Euclidean algorithm would use to find $gcd(f_{n+1}, f_n)$, since $f_{n+1}$ is in $S$.  

19. $A = \frac{1}{I} - 1$. Hence, $A^k = \frac{1}{I} - 1^k$. It follows that $f_{n+1} = f_n + f_{n-1} = \frac{1}{I} - 1^{n+2} + \frac{1}{I} - 1^{n+1}$.  

20. By induction on the number of terms. Basis step: $P(n) = 0$ for $n = 1$, max($-a_1, a_2 - a_1, \ldots, a_n - a_{n-1}, a_n$), which is in $S$ for $n = 2$. Inductive step: Assume that $P(k)$ is true, and if $a_1$, $a_2$, and $a_3$ are in $S$, then $a_1 + a_2 + a_3 = 0$.  

21. By induction on the statement that $a_{n+1} - a_1 = \sum_{k=2}^{n+1} (-1)^k a_k$.  

23. $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$.  

24. a) $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$, $x = 6$, $x = 7$, $x = 8$, $x = 9$.  

b) Let $P(n)$ be the statement that $a_{n+1} = 2a_n$ whenever $(a_n, S) = 0$, and indeed $0 \leq 2 \leq S$. (Inductive step: A sum that $a_n \leq 2k$, whenever $(a_n, S) = 0$, is obtained by $n$ applications of the recursive step. Basis step: $P(0)$ is true, because the only element of $S$ obtained with no applications of the recursive step is $(0, 1)$, and indeed $0 \leq 2 \leq 0$.)  

25. a) The case with 

Then max

be "$f_1 + f_2 + \cdots + f_{n-1} = f_n$". Basis step: $P(1)$ is true because $f_1 = 1 = f_2$. Inductive step: A sum that $f_1 + f_2 + \cdots + f_{n-1} + f_n = f_{n+1}$ is the number of divisions used by the Euclidean algorithm to find $gcd(f_{n+1}, f_n)$.  

27. a) Define $S$ by $(1, 1) \in S$, and if $(a, b) \in S$, then $(a + b, 2) \in S$, $(a, b + 2) \in S$, and $(a + b, b + 1) \in S$. All elements put in $S$ satisfy the condition, because $(1, 1)$ has an even sum of coordinates, and if $(a, b)$ has an even sum of coordinates, then so do $(a + b, 2)$, $(a, b + 2)$, and $(a + b, b + 1)$. Conversely, we show by induction on the sum of the coordinates that if $(a, b) \in S$ is even, then $(a, b) \in S$. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put $(a, b)$ into $S$. Otherwise the sum is at least 4, and at least one of $(a - 2, b)$, $(a, b - 2)$, and $(a - 1, b - 1)$ must have positive integer coordinates whose sum is an even number smaller than $a + b$, and therefore must be in $S$. Then one application of the recursive step shows that $(a, b) \in S$.  

b) Define $S$ by $(1, 1), (1, 2), (2, 1), (2, 2) \in S$, and if $(a, b) \in S$, then $(a + b, 2) \in S$ and $(a, b + 2) \in S$. If $(a, b) \in S$ is odd, then $(a, b) \in S$ and $(a + b, b + 1) \in S$. Conversely we show by induction on the sum of the coordinates that if $(a, b) \in S$ is odd, then $(a, b) \in S$. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put $(a, b)$ into $S$. Otherwise either $a$ or $b$ is at least 3, so at least one of $(a - 2, b)$ and $(a, b - 2)$ must have positive integer coordinates whose sum is smaller than $a + b$, and therefore must be in $S$. Then one application of the recursive step shows that $(a, b) \in S$.  

c) $(1, 6), (2, 3) \in S$ and if $(a, b) \in S$, then $(a + b, 2) \in S$ and $(a, b + 2) \in S$. To prove that our definition works, we note first that $(1, 1), (1, 2)$, and $(2, 1)$ all have an odd coordinate, and if $(a, b)$ has an odd coordinate, then so do $(a + 2, b)$ and $(a, b + 2)$. Conversely, we show by induction on the sum of the coordinates that if $(a, b) \in S$ is odd, then $(a, b) \in S$. If $(a, b) = (1, 1)$, then the basis step put $(a, b)$ into $S$. Otherwise either $a$ or $b$ is at least 3, so at least one of $(a - 2, b)$ and $(a, b - 2)$ must have positive integer coordinates whose sum is smaller than $a + b$, and therefore must be in $S$. Then one application of the recursive step shows that $(a, b) \in S$.  

For sums 5 and 7, the only points are $1, 6$, which the basis step put into $S$, and $2, 3$, which the basis step put into $S$, and $2, 4$, which is in $S$ by one application of the recursive definition. For a sum greater than 7, either $a \geq 3$, or
The recursive step and the basis step of the definition of \(m(st)\) must have positive integer coordinates whose sum is smaller than \(n + 6\) and satisfy the condition for being \(S\). Then one application of the recursive step shows that \((a, b) \in S\). If \(S\) is a set or a variable representing a set, then \(S\) is a well-formed formula. If \(x \in y\) and \(x\) are well-formed formulae, then so are \(\sigma(x, y), (x \cap y),\) and \(x - y\). 33. a) If \(x \subset D\), then \(m(st) = \min(m(st), x)\). b) Let \(m(st) = w\), where \(w \in D^*\) and \(w \in D\). If \(w = \lambda\), then \(m(st) = m(st)\).

Otherwise, \(m(st) = \min(m(st), w, x)\) by the definition of \(m(st)\). Namely, \(m(st) = m(st)\) by the inductive hypothesis of the structural induction, so \(m(st) = m(st)\) by the recursive step and the basis step of the definition of \(m(st)\).

b) Let \(w = w\), where \(w \in D^{*}\) and \(w \in D\). If \(w = \lambda\), then \(m(st) = m(st)\) by the basis step of the definition of \(m(st)\).

34. a) If \(x \subset D\), then \(m(st) = \min(m(st), x)\). b) Let \(m(st) = w\), where \(w \in D^{*}\) and \(w \in D\). If \(w = \lambda\), then \(m(st) = m(st)\).

b) Let \(w = w\), where \(w \in D^{*}\) and \(w \in D\). If \(w = \lambda\), then \(m(st) = m(st)\) by the basis step of the definition of \(m(st)\).

Thus, \(m(st) = m(st)\). Basis step: \(P_{m,n} = \emptyset\) because \(P(\emptyset) = \emptyset\) and \(m(st) = m(st)\) by the recursive step and the basis step of the definition of \(m(st)\).

Inductive step: Assume that for all \(m\neq l\), \(A(m, k) > A(m, l)\). Finally, to find \(A(m, k)\) for all \(m\neq l\), we use the double-induction argument to prove the stronger statement: \(A(m, k) > A(m, l)\) when \(k > l\). Basis step: When \(m = 0\) the statement is true because \(k = l\) implies that \(A(0, k) = 2k > 2l = A(0, l)\), Inductive step: Assume that \(A(m, x) = A(m, y)\) for all nonnegative integers \(x\) and \(y\) with \(x > y\). We will show that this implies that for all nonnegative integers \(m\) and \(k\) such that \(k > l\). Basis step: When \(l = 0\) and \(k > 0\), then \(A(m, 1) = A(m, 0)\).

If \(m = 0\), then \(A(m, 1) = A(m + 1, 1)\). If \(k = 0\), this is greater than \(0\) by the inductive hypothesis. In all cases, \(A(m + 1, k) > 2l + 1\), and in fact, \(A(m, 1) \geq 2l + 1\). If \(l = 1\) and \(k > 1\), then \(A(m, 1) = A(m, 1)\).

For \(m = 1\), the statement is true by the basis step. Hence, by the inductive hypothesis, the claim holds for all nonnegative integers \(m\) and \(k\).

Section 5.4

1. First, we use the recursive step to write \(S^1 = S^1\). Then we use the recursive step to write \(S^2 = S^2\) and so on until \(S^n = S^n\). Finally, we work back through the steps, showing that \(S = S = S\). So \(S = \emptyset\).

2. If \(S\) is ambiguous because \(F(\emptyset) > 0\) and \(F(\emptyset) > 0\), then \(F(\emptyset) > 0\) is not defined because \(F(\emptyset)\) makes no sense. The definition of \(F(\emptyset)\) is ambiguous because both the second and third clause seem to apply. \(F(\emptyset)\) cannot be computed because trying to compute \(F(\emptyset)\) gives \(F(\emptyset) > 0\). So \(F(\emptyset)\) is true for all integers. 59. A The value of \(F(\emptyset)\) is ambiguous.

3. If \(S\) is ambiguous and \(F(\emptyset)\) is not defined because \(F(\emptyset)\) makes no sense. The definition of \(F(\emptyset)\) is ambiguous because both the second and third clause seem to apply. \(F(\emptyset)\) cannot be computed because trying to compute \(F(\emptyset)\) gives \(F(\emptyset) > 0\). So \(F(\emptyset)\) is true for all integers. 59. A The value of \(F(\emptyset)\) is ambiguous.

4. This is not defined because \(F(\emptyset)\) is not defined. \(F(\emptyset)\) is ambiguous and \(F(\emptyset)\) is not defined because \(F(\emptyset)\) makes no sense. The definition of \(F(\emptyset)\) is ambiguous because both the second and third clause seem to apply. \(F(\emptyset)\) cannot be computed because trying to compute \(F(\emptyset)\) gives \(F(\emptyset) > 0\). So \(F(\emptyset)\) is true for all integers. 59. A The value of \(F(\emptyset)\) is ambiguous.