showed that the principle of mathematical induction follows from the well-ordering property.

The other parts of this equivalence are left as Exercises 31, 42, and 43.

**THE WELL-ORDERING PROPERTY**

Every nonempty set of nonnegative integers has a least element.

The well-ordering property can often be used directly in proofs.

**EXAMPLE 5**

Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if \( a \) is an integer and \( d \) is a positive integer, then there are unique integers \( q \) and \( r \) with \( 0 \leq r < d \) and \( a = dq + r \).

**Solution:** Let \( S \) be the set of nonnegative integers of the form \( a - dq \), where \( q \) is an integer. This set is nonempty because \( dq \) can be made as large as desired (taking \( q \) to be a negative integer with large absolute value). By the well-ordering property, \( S \) has a least element \( r = a - dq_0 \).

The integer \( r \) is nonnegative. It is also the case that \( r < d \). If it were not, then there would be a smaller nonnegative element in \( S \), namely, \( a - d(q_0 + 1) \). To see this, suppose that \( r \geq d \). Because \( a = dq_0 + r \), it follows that \( a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0 \). Consequently, there are integers \( q \) and \( r \) with \( 0 \leq r < d \). The proof that \( q \) and \( r \) are unique is left as Exercise 37.

**EXAMPLE 6**

In a round-robin tournament every player plays every other player exactly once and each match has a winner and a loser. We say that the players \( p_1, p_2, \ldots, p_n \) form a cycle if \( p_1 \) beats \( p_2 \), \( p_2 \) beats \( p_3 \), \ldots, \( p_{n-1} \) beats \( p_n \), and \( p_n \) beats \( p_1 \). Use the well-ordering principle to show that if there is a cycle of length \( m \) (\( m \geq 3 \)) among the players in a round-robin tournament, there must be a cycle of three of these players.

**Solution:** We assume that there is no cycle of three players. Because there is at least one cycle in the round-robin tournament, the set of all positive integers \( n \) for which there is a cycle of length \( n \) is nonempty. By the well-ordering property, this set of positive integers has a least element \( k \), which by assumption must be greater than three. Consequently, there exists a cycle of players \( p_1, p_2, p_3, \ldots, p_k \) and no shorter cycle exists.

Because there is no cycle of three players, we know that \( k > 3 \). Consider the first three elements of this cycle, \( p_1, p_2, \) and \( p_3 \). There are two possible outcomes of the match between \( p_1 \) and \( p_2 \). If \( p_2 \) beats \( p_1 \), it follows that \( p_1, p_2, p_3 \) is a cycle of length three, contradicting our assumption that there is no cycle of three players. Consequently, it must be the case that \( p_1 \) beats \( p_2 \). This means that we can omit \( p_2 \) from the cycle \( p_1, p_2, p_3, \ldots, p_k \) to obtain the cycle \( p_1, p_3, p_4, \ldots, p_k \) of length \( k - 1 \), contradicting the assumption that the smallest cycle has length \( k \). We conclude that there must be a cycle of length three.

**Exercises**

1. Use strong induction to show that if you can run one mile or two miles, and if you can always run two more miles once you have run a specified number of miles, then you can run any number of miles.
2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.
3. Let \( P(a) \) be the statement that a postage of \( a \) cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that \( P(a) \) is true for \( a \geq 8 \).
   a) Show that the statements \( P(8), P(9), \) and \( P(10) \) are true, completing the basis step of the proof.
   b) What is the inductive hypothesis of the proof?
   c) What do you need to prove in the inductive step?
   d) Complete the inductive step for \( k \geq 10 \).
   e) Explain why these steps show that this statement is true whenever \( a \geq 8 \).
4. Let \( P(a) \) be the statement that a postage of \( a \) cents can be formed using just 4-cent stamps and 7-cent stamps. The
parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

5. a) Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

6. a) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

8. Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

9. Use strong induction to prove that $\sqrt{2}$ is irrational. (Hint: Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer $b$.)

10. A square that a chocolate bar consists of $n$ squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. A assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into $n$ separate squares. Use strong induction to prove your answer.

11. Consider this variation of the game of Nim. The game begins with $n$ matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer $j$ and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer $j$.

12. Use strong induction to show that every positive integer $n$ can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. (Hint: For the inductive step, separately consider the case where $k = 1$ is even and where it is odd. When it is even, note that $(k−1)/2$ is an integer.)

13. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n−1$ moves are required to assemble a puzzle with $n$ pieces.

14. Suppose you begin with a pile of $n$ stones and split this pile into $p$ stones of one each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have $r$ and $s$ stones in them, respectively, you compute $rs$. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n−1)/2$.

15. Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is square. (Hint: Use strong induction to show that this strategy works. For the first move, the first player should be to chomp the cookie in the bottom row at the far right.)

16. Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is two squares wide, that is, a $2 \times n$ board. (Hint: Use strong induction. The first move of the first player should be to chomp the cookie in the bottom row of the leftmost column.)

17. Use strong induction to show that if a simple polygon with at least four sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon.

18. Use strong induction to show that when a simple polygon $P$ with consecutive vertices $v_1$, $v_2$, ..., $v_n$ is triangulated into $n−2$ triangles, the $n−2$ triangles can be numbered $1$, $2$, ..., $n−2$ so that $v_i$ is a vertex of triangle $i$ for $i = 1, 2, ..., n−2$.

19. Pick's theorem says that the area of a simple polygon $P$ in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P) + B(P)/2 − 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of $P$ and on the boundary of $P$, respectively. Use strong induction on the number of vertices of $P$ to prove Pick's theorem. (Hint: For the basis step, first prove the theorem for rectangles, then for right triangles, and finally for all triangles by noting that the area of a triangle is the area of a larger rectangle containing it with the areas of at most three triangles subtracted. For the inductive step, take advantage of Lemma 1.)
20. Suppose that \( P \) is a simple polygon with vertices \( v_1, v_2, \ldots, v_n \), listed so that consecutive vertices are connected by an edge, and \( v_1 \) and \( v_n \) are connected by an edge. A vertex \( v_i \) is called an ear if the line segment connecting the two vertices adjacent to \( v_i \) is an interior diagonal of the simple polygon. Two ears \( v_i \) and \( v_j \) are called nonoverlapping if the interiors of the triangles with vertices \( v_i \) and its two adjacent vertices and \( v_j \) and its two adjacent vertices do not intersect. Prove that every simple polygon with at least four vertices has at least two nonoverlapping ears.

21. In the proof of Lemma 1 we mentioned that many incorrect methods for finding a vertex \( p \) such that the line segment \( bp \) is an interior diagonal of \( P \) have been published. This exercise presents some of the incorrect ways \( p \) has been chosen in these proofs. Show, by considering one of the polygons drawn here, that for each of these choices of \( p \), the line segment \( bp \) is not necessarily an interior diagonal of \( P 

Exercise 22 and 23 present examples that show inductive loading can be used to prove results in computational geometry.

22. Let \( P(n) \) be the statement that when nonintersecting diagonals are drawn inside a convex polygon with \( n \) sides, at least two vertices of the polygon are not endpoints of any of these diagonals.

a) Show that when we attempt to prove \( P(n) \) for all integers \( n \geq 3 \) using strong induction, the inductive step does not go through.

b) Show that we can prove that \( P(n) \) is true for all integers \( n \geq 3 \) by proving strong induction the stronger assertion \( P(n) \), for \( n = 4 \), where \( P(4) \) states that whenever nonintersecting diagonals are drawn inside a convex polygon with \( n \) sides, at least two nonadjacent vertices are not endpoints of any of these diagonals.

23. Let \( E(n) \) be the statement that in a triangulation of a simple polygon with \( n \) sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon.

a) Explain where a proof using strong induction that \( E(n) \) is true for all integers \( n \geq 4 \) runs into difficulties.

b) Show that we can prove that \( E(n) \) is true for all integers \( n \geq 4 \) by proving by strong induction the stronger statement \( P(n) \) for all integers \( n \geq 4 \), which states that in every triangulation of a simple polygon, at least two of the triangles in the triangulation have two sides bordering the exterior of the polygon.

A vertex \( v \) is called an ear if the line segment connecting the two vertices adjacent to \( v \) is an interior diagonal of \( P \).
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30. Find the flaw with the following "proof" that \( a^n = 1 \) for all nonnegative integers \( n \), wherever \( a \) is a nonzero real number.

\[ a^2 = 1 \] is true by the definition of \( a^2 \).

**Inductive Step:** Assume that \( a^j = 1 \) for all nonnegative integers \( j \) with \( j \leq k \). Then note that \( a^{k+1} = a^k \cdot a^1 = 1 \cdot 1 = 1 \).

31. Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

32. Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

**Basis Step:** We can form postage of three cents using a single four-cent stamp and we can form postage of four cents using a single four-cent stamp.

**Inductive Step:** Assume that we can form postage of \( j \) cents for all nonnegative integers \( j \) with \( j \leq k \) using just three-cent and four-cent stamps. We can then form postage of \( k + 1 \) cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

33. Show that we can prove that \( P(n, k) \) is true for all pairs of positive integers \( n \) and \( k \) if we show

a) \( P(1, 1) \) is true and \( P(n, k) \Rightarrow \{ P(n+1, k) \land P(n, k+1) \} \) is true for all positive integers \( n \) and \( k \).

b) \( P(1, k) \) is true for all positive integers \( k \), and \( P(n, k) \Rightarrow P(n+1, k) \) is true for all positive integers \( n \) and \( k \).

c) \( P(n, 1) \) is true for all positive integers \( n \), and \( P(n, k) \Rightarrow P(n, k+1) \) is true for all positive integers \( n \) and \( k \).

34. Prove that \( \sum_{j=1}^{n} j(j+1)(j+2) \cdots (j+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)(n+k+1)}{(k+1)!} \) for all positive integers \( k \) and \( n \). [Hint: Use a technique from Exercise 33.]

35. Show that if \( a_1, a_2, \ldots, a_n \) are all distinct real numbers, exactly \( n \) - 1 multiplications are used to compute the product of these \( n \) numbers no matter how parentheses are inserted into their product. [Hint: Use strong induction and consider the last multiplication.]

36. The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let \( a \) and \( b \) be positive integers, and let \( S \) be the set of positive integers of the form \( ax + by \), where \( x \) and \( y \) are integers.

a) Show that \( S \) is nonempty.

b) Use the well-ordering property to show that \( S \) has a smallest element \( c \).

c) Show that if \( d \) is a common divisor of \( a \) and \( b \), then \( d \) is a divisor of \( c \).

d) Show that \( c \) divides \( a \). Then \( a = dq + r \), where \( 0 \leq r < c \). Show that \( r \in S \), contradicting the choice of \( c \).

e) Conclude from (c) and (d) that the greatest common divisor of \( a \) and \( b \) exists. Finish the proof by showing that this greatest common divisor is unique.

37. Let \( a \) be an integer and \( d \) be a positive integer. Show that the integers \( x, y \) with \( ax + dy = 0 \) exist, which were shown to exist in Example 5, are unique.

38. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

39. Can you use the well-ordering property to prove the statement: "Every positive integer can be described using no more than fifteen English words?" Assume the words come from a particular dictionary of English. [Hint: Suppose that there are positive integers that cannot be described using no more than fifteen English words. By well ordering, the smallest positive integer that cannot be described using no more than fifteen English words would then exist.]

40. Use the well-ordering principle to show that if \( x \) and \( y \) are real numbers with \( x < y \), then there is a rational number \( r \) with \( x < r < y \). [Hint: Use the Archimedean property, given in Appendix 1, to find a positive integer \( A \) with \( A > 1/(y-x) \). Then show that there is a rational number \( r \) with denominator \( A \) between \( x \) and \( y \) by looking at the numbers \( x, x + 1/A, \ldots, x + A \), where \( j \) is a positive integer.]

41. Show that the well-ordering property can be proved when the principle of mathematical induction is taken as an axiom.

42. Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

43. Show that we can prove the well-ordering property when we take strong induction as an axiom instead of taking the well-ordering property as an axiom.

### 5.3 Recursive Definitions and Structural Induction

#### Introduction

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion**. For instance, the picture in Figure 1 is produced recursively. First, an original picture is given. Then a process of successively superimposing centered smaller pictures on top of the previous pictures is carried out.