Discrete Mathematics

ECS 20 (Fall 2017)

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Homework 1 - For 10/5/2017

Exercise 1

Let A and B be two natural numbers. Follow the proof given below and identify which step(s) is (are) not valid.

<table>
<thead>
<tr>
<th>Step #</th>
<th>Equation</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A = B</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>A × A = B × A</td>
<td>Multiply by B on each side</td>
</tr>
<tr>
<td>3</td>
<td>A² − B² = AB − B²</td>
<td>Subtract B² on each side</td>
</tr>
<tr>
<td>4</td>
<td>(A − B)(A + B) = (A − B)B</td>
<td>Factorize</td>
</tr>
<tr>
<td>5</td>
<td>A + B = B</td>
<td>Simplify: divide by A-B</td>
</tr>
<tr>
<td>6</td>
<td>B + B = B</td>
<td>Base on step 1, A = B, therefore A + B = B + B</td>
</tr>
<tr>
<td>7</td>
<td>2B = B</td>
<td>By definition, B + B = 2B</td>
</tr>
<tr>
<td>8</td>
<td>2 = 1</td>
<td>Simplify: divide by B</td>
</tr>
</tbody>
</table>

There is only one mistake in the proof, in step 5: we cannot divide by A − B as A = B, i.e. A − B = 0!!

Exercise 2

Prove the following statements:

a) The sum of any three consecutive even numbers is always a multiple of 6

Let N be an even number. There exists an integer number k such that n = 2k. The two even numbers that follows N are N + 2 and N + 4, which can be rewritten as 2k + 2 and 2k + 4.
Let $S$ be the sum of these three consecutive even numbers. Then:

\[
S = N + N + 2 + N + 4
\]
\[
= 2k + 2k + 2 + 2k + 4
\]
\[
= 6k + 6
\]
\[
= 6(k + 1)
\]

As $k + 1$ is an integer, $S$ is a multiple of 6. As this is true for all values of $N$, the proposition is always true.

b) The sum of any four consecutive odd numbers is always a multiple of 8

Let $N$ be an odd number. There exists an integer number $k$ such that $N = 2k + 1$. The three odd numbers that follows $N$ are $N + 2$, $N + 4$, and $N + 4$, which can be rewritten as $2k + 3$, $2k + 5$ and $2k + 7$. Let $S$ be the sum of these four consecutive odd numbers. Then:

\[
S = 2k + 1 + 2k + 3 + 2k + 5 + 2k + 7
\]
\[
= 8k + 16
\]
\[
= 8(k + 2)
\]

As $k + 2$ is an integer, $S$ is a multiple of 8. As this is true for all values of $N$, the proposition is always true.

c) Prove that if you add the squares of two consecutive integer numbers and then add one, you always get an even number.

Let $N$ be an integer number. The number that follows $N$ is $N + 1$. Let $S$ be the sum of the squares of these two consecutive numbers. Then:

\[
S = N^2 + (N + 1)^2
\]
\[
= N^2 + N^2 + 2N + 1
\]
\[
= 2N^2 + 2N + 1
\]

Therefore,

\[
S + 1 = 2(N^2 + N + 1)
\]

As $(N^2 + N + 1)$ is an integer, $S + 1$ is a multiple of 2, i.e. an even number. As this is true for all values of $N$, the proposition is always true.

**Exercise 3**

Let $x$ be a real number. Solve the equation $3^{2x} - 2(3^x) + 1 = 0$.

Solution: Let $x$ be a real number. Let us define $P(x) = 3^{2x} - 2(3^x) + 1$. We simplify $P(x)$ ::

\[
P(x) = 3^{2x} - 2(3^x) + 1
\]
\[
= (3^x)^2 - 2(3^x) + 1
\]
Let us define \( y = 3^x \). Substituting in the equation above, we get:

\[
P(x) = y^2 - 2y + 1
= (y - 1)^2
\]

Solving \( P(x) = 0 \) is therefore equivalent to solving \((y - 1)^2 = 0\), which has only one solution, \( y = 1 \). Therefore

\[
(3^x) = 1
\]

Taking the \( \text{Log} \) of this equation:

\[
x \text{Log}(3) = 0
\]

Therefore \( x = 0 \).

Substituting back into \( P(x) \): \( P(0) = 3^0 - 2 \times 3^0 + 1 = 1 - 2 + 1 = 0 \).

**Exercise 4**

Prove the following identities for \( p, q, m, n, x, \) and \( y \) real numbers:

a) \( 8(p - q) + 4(p + q) = 2(p + 3q) + 10(p - q) \)

Let \( p \) and \( q \) be two real numbers, and let \( LHS = 8(p - q) + 4(p + q) \) and \( RHS = 2(p + 3q) + 10(p - q) \). Then:

\[
LHS = 8p - 8q + 4p + 4q
= 12p - 4q
\]

and

\[
RHS = 2p + 6q + 10p - 10q
= 12p - 4q
\]

Therefore \( LHS = RHS \) for all \( p \) and \( q \), and the identity is true.

b) \( x(m - n) + y(n + m) = m(x + y) + n(y - x) \)

Let \( x, y, m \) and \( n \) be four real numbers, and let \( LHS = x(m - n) + y(n + m) \) and \( RHS = m(x + y) + n(y - x) \). Then:

\[
LHS = xm - xn + yn + ym
\]

and

\[
RHS = xm - xn + ym + yn
\]

Therefore \( LHS = RHS \) for all \( x, y, n \) and \( m \), and the identity is true.
c) \((x + 3)(x + 8) - (x - 6)(x - 4) = 21x\)

Let \(x\) be a real number and let \(LHS = (x + 3)(x + 8) - (x - 6)(x - 4)\) and \(RHS = 21x\). Then:

\[
LHS = x^2 + 8x + 3x + 24 - x^2 + 4x + 6x - 24 = 21x = RHS
\]

The identity is true for all \(x\).

d) \(m^8 - 1 = (m^2 - 1)(m^2 + 1)(m^4 + 1)\)

Let \(m\) be a real number and let \(LHS = m^8 - 1\) and \(RHS = (m^2 - 1)(m^2 + 1)(m^4 + 1)\). Then

\[
LHS = (m^4)^2 - 1^2 = (m^4 - 1)(m^4 + 1) = ((m^2)^2 - 1)(m^4 + 1) = (m^2 - 1)(m^2 + 1)(m^4 + 1) = RHS
\]

The identity is true for all \(m\).

**Extra credit**

Let us consider a floor covered with square tiles; all the tiles have the same size, with Width=Height=A. We drop a coin of radius R on the floor. What is the probability that the coin falls on top of (at least) one line that delimits a tile? We assume such a line to be infinitesimally thin.

Let us draw a figure so that we have visual representation of the problem:

![Figure 1: A representation of the floor](image)

We only need to consider one tile. When a coin falls in that tile, there are three situations to consider:
i) The coin falls on one of the line (type 1 on the figure); the center of that coin must be within a distance \( R \) from the line

ii) The coin just touches the line (type 2 on the figure); the center of that coin is at distance \( R \) from the line

iii) The coin does not touch a line (type 3 on the figure); the center of that coin is inside the red square, at a distance at least \( R \) from a line.

The probability \( P_{NT} \) that the coin does not touch the line is therefore the ratio of the area of the red square and the area of a tile.

Area of a tile: \( A_T = D^2 \)
Area of the red square: \( A_R = (D - 2R)^2 \)
Therefore:

\[
P_{NT} = \frac{(D - 2R)^2}{D^2}
\]

and the probability \( P_T \) that the coin touches a line is:

\[
P_T = 1 - P_{NT} = \frac{4RD - 4R^2}{D^2}
\]