Homework 1 - For 1/15/2019

Exercise 1

Let A and B be two natural numbers. Follow the proof given below and identify which step(s) is (are) not valid.

<table>
<thead>
<tr>
<th>Step #</th>
<th>Equation</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A = B$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>$A \times A = B \times A$</td>
<td>Multiply by B on each side</td>
</tr>
<tr>
<td>3</td>
<td>$A^2 - B^2 = AB - B^2$</td>
<td>Subtract $B^2$ on each side</td>
</tr>
<tr>
<td>4</td>
<td>$(A - B)(A + B) = (A - B)B$</td>
<td>Factorize</td>
</tr>
<tr>
<td>5</td>
<td>$A + B = B$</td>
<td>Simplify: divide by $A-B$</td>
</tr>
<tr>
<td>6</td>
<td>$B + B = B$</td>
<td>Base on step 1, $A = B$, therefore $A + B = B + B$</td>
</tr>
<tr>
<td>7</td>
<td>$2B = B$</td>
<td>By definition, $B + B = 2B$</td>
</tr>
<tr>
<td>8</td>
<td>$2 = 1$</td>
<td>Simplify: divide by $B$</td>
</tr>
</tbody>
</table>

There is only one mistake in the proof, in step 5: we cannot divide by $A - B$ as $A = B$, i.e. $A - B = 0$!!

Exercise 2

Prove the following statements:

a) The sum of any three consecutive even numbers is always a multiple of 6

Let $N$ be an odd number. There exists an integer number $k$ such that $n = 2k + 1$. The two odd numbers that follows $N$ are $N + 2$ and $N + 4$, which can be rewritten as $2k + 3$ and...
Let $S$ be the sum of these three consecutive odd numbers. Then:

\[
S = N + N + 2 + N + 4 \\
   = 2k + 1 + 2k + 3 + 2k + 5 \\
   = 6k + 9 \\
   = 3(2k + 3)
\]

As $2k + 3$ is an integer, $S$ is a multiple of 3. As this is true for all values of $N$, the proposition is always true.

b) **The sum of any four consecutive odd numbers is always a multiple of 8**

Let $N$ be an odd number. There exists an integer number $k$ such that $N = 2k + 1$. The three odd numbers that follows $N$ are $N + 2$, $N + 4$, and $N + 4$, which can be rewritten as $2k + 3$, $2k + 5$ and $2k + 7$. Let $S$ be the sum of these four consecutive odd numbers. Then:

\[
S = 2k + 1 + 2k + 3 + 2k + 5 + 2k + 7 \\
   = 8k + 16 \\
   = 8(k + 2)
\]

As $k + 2$ is an integer, $S$ is a multiple of 8. As this is true for all values of $N$, the proposition is always true.

c) **Prove that if you add the squares of two consecutive integer numbers and then add one, you always get an even number.**

Let $N$ be an integer number. The number that follows $N$ is $N + 1$. Let $S$ be the sum of the squares of these two consecutive numbers. Then:

\[
S = N^2 + (N + 1)^2 \\
   = N^2 + N^2 + 2N + 1 \\
   = 2N^2 + 2N + 1
\]

Therefore,

\[
S + 1 = 2(N^2 + N + 1)
\]

As $(N^2 + N + 1)$ is an integer, $S + 1$ is a multiple of 2, i.e. an even number. As this is true for all values of $N$, the proposition is always true.

**Exercise 3**

Let $x$ be a real number. Solve the equation \(5^{2x} - 2(5^x) + 1 = 0\).

Solution: Let $x$ be a real number. Let us define $P(x) = 5^{2x} - 2(5^x) + 1$. We simplify $P(x)$:

\[
P(x) = 5^{2x} - 2(5^x) + 1 \\
     = (5^x)^2 - 2(5^x) + 1
\]
Let us define \( y = 5^x \). Substituting in the equation above, we get:

\[
P(x) = y^2 - 2y + 1 = (y - 1)^2
\]

Solving \( P(x) = 0 \) is therefore equivalent to solving \( (y - 1)^2 = 0 \), which has only one solution, \( y = 1 \). Therefore

\[
(5^x) = 1
\]

Taking the Log of this equation:

\[
x \log(5) = 0
\]

Therefore \( x = 0 \).

Substituting back into \( P(x) \): \( P(0) = 5^0 - 2 \times 5^0 + 1 = 1 - 2 + 1 = 0 \).

**Exercise 4**

*Prove the following identities for \( p, q, m, n, x, \) and \( y \) real numbers:*

a) \( 8(p - q) + 4(p + q) = 2(p + 3q) + 10(p - q) \)

Let \( p \) and \( q \) be two real numbers, and let \( LHS = 8(p - q) + 4(p + q) \) and \( RHS = 2(p + 3q) + 10(p - q) \). Then:

\[
LHS = 8p - 8q + 4p + 4q = 12p - 4q
\]

and

\[
RHS = 2p + 6q + 10p - 10q = 12p - 4q
\]

Therefore \( LHS = RHS \) for all \( p \) and \( q \), and the identity is true.

b) \( x(m - n) + y(n + m) = m(x + y) + n(y - x) \)

Let \( x, y, m \) and \( n \) be four real numbers, and let \( LHS = x(m - n) + y(n + m) \) and \( RHS = m(x + y) + n(y - x) \). Then:

\[
LHS = xm - xn + yn + ym
\]

and

\[
RHS = xm - xn + ym + yn
\]

Therefore \( LHS = RHS \) for all \( x, y, n \) and \( m \), and the identity is true.
c) \((x + 3)(x + 8) - (x - 6)(x - 4) = 21x\)

Let \(x\) be a real number and let \(\text{LHS} = (x + 3)(x + 8) - (x - 6)(x - 4)\) and \(\text{RHS} = 21x\). Then:

\[
\text{LHS} = x^2 + 8x + 3x + 24 - x^2 + 4x - 24 \\
= 21x \\
= \text{RHS}
\]

The identity is true for all \(x\).

d) \(m^8 - 1 = (m^2 - 1)(m^2 + 1)(m^4 + 1)\)

Let \(m\) be a real number and let \(\text{LHS} = m^8 - 1\) and \(\text{RHS} = (m^2 - 1)(m^2 + 1)(m^4 + 1)\). Then

\[
\text{LHS} = (m^4)^2 - 1^2 \\
= (m^4 - 1)(m^4 + 1) \\
= ((m^2)^2 - 1)(m^4 + 1) \\
= (m^2 - 1)(m^2 + 1)(m^4 + 1) \\
= \text{RHS}
\]

The identity is true for all \(m\).

**Extra credit**

*Prove that if you add the cubes of two consecutive integer numbers and then add one, you always get an even number.*

Let \(N\) be an integer number. The number that follows \(N\) is \(N + 1\). Let \(S\) be the sum of the cubes of these two consecutive numbers. Then:

\[
S = N^3 + (N + 1)^3 \\
= N^3 + N^3 + 3N^2 + 3N + 1 \\
= 2N^3 + 3N(N + 1) + 1
\]

Therefore,

\[
S + 1 = 2(N^3 + 1) + 3N(N + 1)
\]

Let us prove now that if \(N\) is an integer, then \(N(N + 1)\) is even.

Proof: \(N\) is an integer. We look at two cases:

a) If \(N\) is even, there exists an integer \(k\) such that \(N = 2k\). Then \(N(N + 1) = 2k(2k + 1)\). Since \(k(2k+1)\) is an integer, \(N(N + 1)\) is even.

b) If \(N\) is odd, there exists an integer \(k\) such that \(N = 2k + 1\). Then \(N(N + 1) = 2(k+1)(2k+1)\).

Since \((k+1)(2k+1)\) is an integer, \(N(N + 1)\) is even.

Therefore \(N(N + 1)\) is even for all integer numbers \(N\). There exists an integer \(k\) such that \(N(N + 1) = 2k\). Then,

\[
S + 1 = 2(N^3 + 1) + 6k \\
= 2(N^3 + 3k + 1)
\]

As \((N^3 + 3k + 1)\) is an integer, \(S + 1\) is a multiple of 2, i.e. an even number. As this is true for all values of \(N\), the proposition is always true.