Homework 3: Solutions

ECS 20 (Fall 2016)

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Exercise 1

Show that this implication is a tautology, by using a table of truth: \[(p ∨ q) ∧ (p → r) ∧ (q → r)] \rightarrow r.\]

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<th>r</th>
<th>p ∨ q</th>
<th>p → r</th>
<th>q → r</th>
<th>A: [(p ∨ q) ∧ (p → r) ∧ (q → r)]</th>
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Exercise 2

Show that \[(p ∨ q) ∧ (¬p ∨ r) \rightarrow (q ∨ r)\] is a tautology

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<th>r</th>
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<th>¬p ∨ r</th>
<th>A: (p ∨ q) ∧ (¬p ∨ r)</th>
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Exercise 3

a) Let \(x\) be a real number. Show that “if \(x^2\) is irrational, it follows that \(x\) is irrational.”
Let $p : x^2$ is irrational, and let $q : x$ is irrational. We need to prove that $p \rightarrow q$. We use an indirect proof, i.e. we show that $\neg q \rightarrow \neg p$.

Let us assume $\neg q$, i.e. $x$ is rational. There exists an integer $a$ and a non-zero integer $b$ such that $x = \frac{a}{b}$. Then $x^2 = \frac{a^2}{b^2}$. Since $a^2$ and $b^2$ are integers, $x^2$ is a rational number. Therefore $\neg p$ is true. Therefore $\neg q \rightarrow \neg p$ is true, and consequently $p \rightarrow q$ is true.

b) Based on question a), can you say that “if $x$ is irrational, it follows that $x^2$ is irrational.”

It is not a valid argument. The statement in a) can be simplified as “$p \rightarrow q$, while the second statement is the converse of the first statement: they are not equivalent.

**Exercise 4**

Prove that a square of an integer ends with a 0, 1, 4, 5 6 or 9. (Hint: let $n = 10k + l$, where $l = 0, 1, 9$)

Let $n$ be an integer; there exists two integers $k$ and $l$ such that $n = 10k + l$ where $0 \leq l \leq 9$. We get:

$$n^2 = (10k + l)^2 = 100k + 20kl + l^2 = k \times 100 + 2kl \times 10 + l^2$$

$k \times 100$ and $2kl \times 10$ are multiples of 10. Therefore, $n^2$ ends as $l^2$. In the following table, we show that $l^2$ always end with a 0, 1, 4, 5, 6, or 9.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$l^2$</th>
<th>end</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>8</td>
<td>64</td>
<td>4</td>
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<tr>
<td>9</td>
<td>81</td>
<td>1</td>
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</table>

**Exercise 5**

Prove that if $n$ is a positive integer, then $n$ is even if and only if $5n + 6$ is even.

Let $p$ be the proposition “$n$ is even” and $q$ be the proposition “$5n + 6$ is even”. We want to show that $p \leftrightarrow q$, which is logically equivalent to show that $p \rightarrow q$ and $q \rightarrow p$.

i) Let us show $p \rightarrow q$:

Hypothesis: $p$ is true, i.e. $n$ is even. As $n$ is even, there exists an integer $k$ such that $n = 2k$. We get:
\[
5n + 6 = 5(2k) + 6 \\
= 10k + 6 \\
= 2 \times (5k + 3)
\]

Since \(5k + 3\) is an integer, \(5n + 6\) is a multiple of 2: it is even.

ii) Let us show \(q \rightarrow p\):
Hypothesis: \(q\) is true, i.e. \(5n + 6\) is even. As \(5n + 6\) is even, there exists an integer \(k\) such that \(5n + 6 = 2k\). We get:

\[
5n = 2k - 6 \\
n = 2k - 6 - 4n \\
n = 2 \times (k - 3 - 2n)
\]

Since \(k - 3 - 2n\) is an integer, \(n\) is a multiple of 2: it is even.

We conclude: \(n\) is even \(\iff\) \(5n + 6\) is even.

**Exercise 6**

Prove that either \(3 \times 100450 + 15\) or \(3 \times 100450 + 16\) is not a perfect square.

Let \(n = 3 \times 100450 + 15\). The two numbers are \(n\) and \(n + 1\).

Proof by contradiction: Let us suppose that both \(n\) and \(n + 1\) are perfect squares:

\[
\exists k \in \mathbb{Z}, k^2 = n \\
\exists l \in \mathbb{Z}, l^2 = n + 1
\]

Then

\[
l^2 = k^2 + 1 \\
(l - k)(l + k) = 1
\]

Since \(l\) and \(k\) are integers, there are only two cases:

- \(l - k = 1\) and \(l + k = 1\), i.e. \(l = 1\) and \(k = 0\). Then we would have \(k^2 = 0\), i.e. \(n = 0\): contradiction

- \(l - k = -1\) and \(l + k = -1\), i.e. \(l = -1\) and \(k = 0\). Again, contradiction.

We can conclude that the proposition is true.

**Exercise 7**

Prove or disprove that if \(a\) and \(b\) are rational numbers, then \(a^b\) is also rational.

It is not true. Let \(a = 2\) and \(b = 1/2\), both \(a\) and \(b\) are rational numbers. However, \(a^b = 2^{1/2} = \sqrt{2}\) which is not rational (see lecture notes).
Exercise 8

Prove that at least one of the real numbers \(a_1, a_2, \ldots, a_n\) is greater than or equal to the average of these numbers. What kind of proof did you use?

We use a proof by contradiction.

Suppose none of the real numbers \(a_1, a_2, \ldots, a_n\) is greater than or equal to the average of these numbers, denoted by \(\bar{a}\).

By definition

\[
\bar{a} = \frac{a_1 + a_2 + \ldots + a_n}{n}
\]

Our hypothesis is that:

\[
\begin{align*}
    a_1 &< \bar{a} \\
    a_2 &< \bar{a} \\
    \ldots &< \ldots \\
    a_n &< \bar{a}
\end{align*}
\]

We sum up all these equations and get the following:

\[
a_1 + a_2 + \ldots + a_n < n \times \bar{a}
\]

Replacing \(\bar{a}\) in equation (9) by its value given in equation (4) we get:

\[
a_1 + a_2 + \ldots + a_n < a_1 + a_2 + \ldots + a_n
\]

This is not possible: a number cannot be strictly smaller than itself: we have reached a contradiction. Therefore our hypothesis was wrong, and the original statement was correct.

Exercise 9

The correct order is: 3, 5, 4, 2, 1.

Exercise 10

Prove that these four statements are equivalent: (i) \(n^2\) is odd, (ii) \(1 - n\) is even, (iii) \(n^3\) is odd, (iv) \(n^2 + 1\) is even.

Let us define the four propositions:

- \(p\) : \(n^2\) is odd
- \(q\) : \(1 - n\) is even
- \(r\) : \(n^3\) is odd
- \(s\) : \(n^2 + 1\) is even
we will show:

- $q \leftrightarrow p$
- $q \leftrightarrow r$
- $q \leftrightarrow s$

If these three logical equivalence are true, all four propositions are equivalent.

1) **Proof 1**: $1 - n$ is even $\leftrightarrow n^2$ is odd.

We need to show two implications: (1) $1 - n$ is even implies $n^2$ is odd and (2), $n^2$ is odd implies that $1 - n$ is even.

a) **Implication 1**: $q \rightarrow p$

We use a direct proof.

Hypothesis: $q$ is true, i.e. $1 - n$ is even. There exists an integer $k$ such that $1 - n = 2k$. Therefore $n = 1 - 2k$. Taking the squares on each side, we get:

$$n^2 = (1 - 2k)^2 = 4k^2 - 2k + 1 = 2(2k^2 - k) + 1$$

Therefore $n^2$ is odd. We conclude that $q \rightarrow p$.

b) **Implication 2**: $p \rightarrow q$.

We use an indirect proof, i.e. we show that: $\neg q \rightarrow \neg p$.

* $\neg q$: $1 - n$ is odd
* $\neg p$: $n^2$ is even.

Let us suppose that $1 - n$ is odd. There exists an integer $k$ such that $1 - n = 2k + 1$; therefore $n = -2k$. Taking the square, we find that $n^2 = 4k^2$, and therefore $n^2$ is even. We conclude that $\neg q \rightarrow \neg p$; its contrapositive is then also true, i.e. $p \rightarrow q$.

We conclude that $q \rightarrow p$ and $p \rightarrow q$, and therefore $p \leftrightarrow q$.

2) **Proof 2**: $1 - n$ is even $\leftrightarrow n^3$ is odd.

We need to show two implications: (1) $1 - n$ is even implies $n^2$ is odd and (2), $n^2$ is odd implies that $1 - n$ is even.

a) **Implication 1**: $q \rightarrow r$

We use a direct proof.

Hypothesis: $q$ is true, i.e. $1 - n$ is even. There exists an integer $k$ such that $1 - n = 2k$. Therefore $n = 1 - 2k$. Taking the cubes on each side, we get:

$$n^3 = (1 - 2k)^3 = -8k^3 + 12k^2 - 6k + 1 = 2(-4k^3 + 6k^2 - 3k) + 1$$

Therefore $n^3$ is odd. We conclude that $q \rightarrow r$.

b) **Implication 2**: $r \rightarrow q$.

We use an indirect proof, i.e. we show that: $\neg q \rightarrow \neg r$.

* $\neg q$: $1 - n$ is odd
* $\neg p$: $n^3$ is even.
Let us suppose that $1 - n$ is odd. There exists an integer $k$ such that $1 - n = 2k + 1$; therefore $n = -2k$. Taking the cube, we find that $n^3 = 8k^3 = 2(4k^3)$, and therefore $n^3$ is even.

We conclude that $\neg q \rightarrow \neg r$; its contrapositive is then also true, i.e. $r \rightarrow q$.

We conclude that $q \rightarrow r$ and $r \rightarrow q$, and therefore $r \Leftrightarrow q$.

1) Proof 3: $1 - n$ is even $\iff n^2 + 1$ is even.

We need to show two implications: (1) $1 - n$ is even implies $n^2 + 1$ is even and (2), $n^2 + 1$ is even implies that $1 - n$ is even.

This is nearly a copy of proof 1!!

a) Implication 1: $q \rightarrow s$
We use a direct proof.
Hypothesis: $q$ is true, i.e. $1 - n$ is even. There exists an integer $k$ such that $1 - n = 2k$.
Therefore $n = 1 - 2k$. Taking the squares on each side, we get:

$$n^2 = (1 - 2k)^2 = 4k^2 - 2k + 1 = 2(2k^2 - k) + 1$$

Therefore:

$$n^2 + 1 = 2(2k^2 - k) + 1 + 1 = 2 \ast (2k^2 - k + 1)$$

Therefore $n^2 + 1$ is even. We conclude that $q \rightarrow s$.

b) Implication 2: $s \rightarrow q$.
We use an indirect proof, i.e. we show that: $\neg q \rightarrow \neg s$.
* $\neg q$: $1 - n$ is odd
* $\neg s$: $n^2 + 1$ is odd.

Let us suppose that $1 - n$ is odd. There exists an integer $k$ such that $1 - n = 2k + 1$; therefore $n = -2k$. Taking the square, we find that $n^2 = 4k^2$, and therefore $n^2 + 1 = 4k^2 + 1$, i.e. $n^2 + 1$ is odd.
We conclude that $\neg q \rightarrow \neg s$; its contrapositive is then also true, i.e. $s \rightarrow q$.

We conclude that $q \rightarrow s$ and $p \rightarrow s$, and therefore $s \Leftrightarrow q$.

**Extra Credit**

Use Exercise 8 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

Let $a_1, a_2, ..., a_{10}$ be an arbitrary order of 10 positive integers from 1 to 10 being placed around a circle:

Since the ten numbers $a$ correspond to the first 10 positive integers, we get:

$$a_1 + a_2 + ... + a_{10} = 1 + 2 + ... + 10 = 55$$ (1)
Notice that the $a_1, a_2, \ldots, a_{10}$ are not necessarily in the order 1, 2, ..., 10. They do include however the ten integers from 1 to 10: these is why the sum is 55.

Let us now consider the different sums $S_i$ of three consecutive sites around the circle. There are 10 such sums:

$$
S_1 = a_1 + a_2 + a_3 \\
S_2 = a_2 + a_3 + a_4 \\
S_3 = a_3 + a_4 + a_5 \\
S_4 = a_4 + a_5 + a_6 \\
S_5 = a_5 + a_6 + a_7 \\
S_6 = a_6 + a_7 + a_8 \\
S_7 = a_7 + a_8 + a_9 \\
S_8 = a_8 + a_9 + a_{10} \\
S_9 = a_9 + a_{10} + a_1 \\
S_{10} = a_{10} + a_1 + a_2
$$

We do not know the values of the individual sums $S_i$; however, we can compute the sum of these numbers:

$$
S_1 + S_2 + \ldots + S_{10} = (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + \ldots + (a_{10} + a_1 + a_2) \\
= 3 \times (a_1 + a_2 + \ldots + a_{10}) \\
= 3 \times 55 \\
= 165
$$

The average of $S_1, S_2, \ldots, S_{10}$ is therefore:
\[
\bar{S} = \frac{S_1 + S_2 + ... + S_{10}}{10} \\
= \frac{165}{10} \\
= 16.5
\]

Based on the conclusion of Exercise 8, at least one of \( S_1, S_2, ..., S_{10} \) is greater to or equal to \( \bar{S} \), i.e., 16.5. Because \( S_1, S_2, ..., S_{10} \) are all integers, they cannot be equal to 16.5. Thus, at least one of \( S_1, S_2, ..., S_{10} \) is greater to or equal to 17.