Exercise 1

Show that this implication is a tautology, by using a truth table:

\[ (p \lor q) \land (p \rightarrow r) \land (q \rightarrow r) \rightarrow r \]

Exercise 2

Show that \( (p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r) \) is a tautology

Exercise 3

a) Let \( x \) be a real number. Show that if \( x^2 \) is irrational, then \( x \) is irrational.

b) Based on question a), can you say that “if \( x \) is irrational, it follows that \( x^2 \) is irrational.”?

Exercise 4:

Prove that a square of an integer ends with a 0, 1, 4, 5 6 or 9. (Hint: let \( n = 10k+l \), where \( l = 0, 1, \ldots, 9 \))

Exercise 5:

Prove that if \( n \) is a positive integer, then \( n \) is even if and only if \( 5n+6 \) is even.

Exercise 6:

Prove that either \( 3x10^{450} + 15 \) or \( 3x10^{450} + 16 \) is not a perfect square. Is your proof constructive, or non-constructive?

Exercise 7:

Prove or disprove that if \( a \) and \( b \) are rational numbers, then \( a^b \) is also rational.

Exercise 8:

Prove that at least one of the real numbers \( a_1, a_2, \ldots, a_n \) is greater than or equal to the average of these numbers. What kind of proof did you use?

Exercise 9:

The proof below has been scrambled. Please put it back in the correct order.
**Claim:** For all \( n \geq 9 \), if \( n \) is a perfect square, then \( n-1 \) is not prime.

Since \((n-1)\) is the product of 2 integers greater than 1, we know \((n-1)\) is not prime \( (1) \)

Since \( m \geq 3 \), it follows that \( m-1 \geq 2 \) and \( m+1 \geq 4 \) \( (2) \)

Let \( n \) be a perfect square such that \( n \geq 9 \) \( (3) \)

This means that \( n-1 = m^2-1 = (m-1)(m+1) \) \( (4) \)

There is an integer \( m \geq 3 \) such that \( n=m^2 \) \( (5) \)

**Exercise 10**
Prove that these four statements are equivalent: (i) \( n^2 \) is odd, (ii) \( 1-n \) is even, (iii) \( n^3 \) is odd, (iv) \( n^2+1 \) is even.

**Extra credit:**
Use Exercise 8 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.