Homework 9 (optional: won’t be graded)

ECS 20 (Winter 2019)

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Exercise 1: 10 points

Using induction, show that \( \forall n \in \mathbb{N}, \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \).

Exercise 2: 10 points

Using induction, show that \( \forall n \in \mathbb{N}, \sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4} \).

Exercise 3: 10 points

Show that \( \forall n \in \mathbb{N}, n > 1, \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n} \).

Exercise 4: 10 points

Show that \( \forall n \in \mathbb{N}, n > 3, n^2 - 7n + 12 \geq 0 \).

Exercise 5: 10 points

Show that \( \forall n \in \mathbb{N}, n > 1, \) a set \( S_n \) with \( n \) elements has \( \frac{n(n-1)}{2} \) subsets that contain exactly two elements.

Exercise 6: 10 points

Find the flaw with the following proof that : \( P(n) : a^n = 1 \) for all non negative integer \( n \), whenever \( a \) is a non zero real number:

- Basis step: \( P(0) \) is true: \( a^0 = 1 \) is true, by definition of \( a^0 \)
- **Strong Inductive step**: assume that \( a^j = 1 \) for all non negative integers \( j \) with \( j \leq k \). Then note that:

\[
a^{k+1} = \frac{a^k a^k}{a^{k-1}} = \frac{1 \times 1}{1} = 1
\]

Therefore \( P(k + 1) \) is true.

The principle of proof by strong mathematical induction allows us to conclude that \( P(n) \) is true for all \( n \geq 0 \).

**Exercise 7: 10 points**

Show that \( \forall n \in \mathbb{N}, \) 21 divides \( 4^{n+1} + 5^{2n-1} \).

**Exercise 8: 10 points**

Show that \( \forall n \in \mathbb{N}, f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1} \) where \( f_n \) are the Fibonacci numbers.

**Exercise 9: 10 points**

Show that \( \forall n \in \mathbb{N}, f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1 \) where \( f_n \) are the Fibonacci numbers.

**Extra Credit: 5 points**

Show that \( \forall n \in \mathbb{N}, n > 1, \) a set \( S_n \) with \( n \) elements has \( \frac{n(n-1)(n-2)}{6} \) subsets that contain exactly three elements.