Homework 5: due 2/12/2019

ECS 20 (Winter 2019)

Patrice Koehl
koehl@cs.ucdavis.edu

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Exercise 1 (10 points for each subquestion: 20 total)

a) Show that the following statement is true: "If there exist two integers \( n \) and \( m \) such that \( 2n^2 + 2n + 1 = 2m \), then \( 2n = 3 \).

b) If \( x \) and \( y \) are rational numbers such that \( x < y \), show that there exists a rational number \( z \) with \( x < z < y \).

Exercise 2 (10 points)

Let \( x \) be a real number. Show that \( \lfloor \frac{x}{3} \rfloor + \lfloor \frac{x+1}{3} \rfloor + \lfloor \frac{x+2}{3} \rfloor = \lfloor x \rfloor \).

Exercise 3 (10 points)

This is a generalization of exercise 2:
Let \( x \) be a real number and \( N \) an integer greater or equal to 3.
Show that \( \lfloor x \rfloor = \lfloor \frac{x}{N} \rfloor + \lfloor \frac{x+1}{N} \rfloor + \ldots + \lfloor \frac{x+N-1}{N} \rfloor \).

(Hint: instead of following a proof similar to the one you used for exercise 2, define:
\[
f(x) = \lfloor x \rfloor - \lfloor \frac{x}{N} \rfloor - \lfloor \frac{x+1}{N} \rfloor - \ldots - \lfloor \frac{x+N-1}{N} \rfloor
\]
and show that \( f(x) \) is periodic, with period 1.)

Exercise 4 (10 points)

Let \( x \) be a real number. Then show that \((\lfloor x \rfloor - x)(x - \lfloor x \rfloor) \leq \frac{1}{4}\)

Exercise 5 (10 points for each subquestion: 20 total)

Let \( x \) be a real number. Solve the following equations:

a) \( \lfloor x^2 + x - 5 \rfloor = \frac{1}{2}x \)

b) \( 2\lfloor 4 - x \rfloor = 2x + 1 \)
Extra Credit \textit{(5 points)}

Let \( x \) and \( y \) be two real numbers such that \( 0 < x \leq y \). We define:

a) The customized arithmetic mean \( m \) of \( x \) and \( y \): \( m = \frac{x + 2y}{3} \)

b) The customized geometric mean \( g \) of \( x \) and \( y \): \( g = \sqrt[3]{x^2y^5} \)

c) The customized harmonic mean \( h \) of \( x \) and \( y \): \( \frac{3}{h} = \left( \frac{1}{x} + \frac{2}{y} \right) \)

Show that:

\[ x \leq h \leq g \leq m \leq y \]