5. Sets

1.1 Definition  A set is an unacknowledged collection of objects.

Examples
- Set \(S\) of all odd numbers between 0 and 10
  \[S = \{1, 3, 5, 7, 9\}\]
- Days of the week
  \[W = \{M, T, W, Th, F, S, S\}\]
- Sets of numbers:
  \[\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\]

1.2. How to describe a set

A set is well-defined if there is a clear rule that defines membership.

Let us consider the set \(S\) of all integers between 2 and 40.

There are two methods to describe \(S\) symbolically:

\[\{n | 2 \leq n \leq 40\}\]
The roster notation is a complete, implied listing of all elements of $S$:

$$S = \{2, 4, 6, 8, \ldots, 40\}$$

The set-builder notation is a symbolic representation used when the roster notation is cumbersome:

$$S = \{x \mid x \in \mathbb{N} \text{ and } 2 \leq x \leq 40 \text{ and } \text{even}\}$$

There is an additional method that is useful to get intuition about set: the Venn diagram.

1.3 Paradoxes

The objects, or elements, of a set do not have to be numbers. We can even build sets of sets. This leads to some paradoxes:

The barber's paradox. Suppose there is a town with one barber, and all men keep clean shaven. Let $S$ be the set of men that are shaved by the barber. Can we define $S$ with: "The barber shaves all, and only those men who do not shave themselves"? Does the barber belong to $S$? -> paradox
Russell's paradox

Let \( M \) be the set of all sets that do not contain themselves as members.

\[
M = \{ A \mid A \not\in A \}
\]

\( M \) seems to be well defined, i.e. there is an explicit rule that defines membership. However ... does \( M \) belong to \( M \)?

- If \( M \in M \), then \( M \not\in M \) [paradox!]
- If \( M \not\in M \), then \( M \) should belong to \( M \)

We will leave this to logicians.

### 1.4 Terminology

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Example</th>
<th>Reads as</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \in )</td>
<td>element of</td>
<td>( x \in A )</td>
<td>( x ) is an element of ( A )</td>
</tr>
<tr>
<td>( \subseteq )</td>
<td>subset</td>
<td>( A \subseteq B )</td>
<td>( A ) is a subset of ( B )</td>
</tr>
<tr>
<td>( \cup )</td>
<td>union</td>
<td>( A \cup B )</td>
<td>( A ) union ( B )</td>
</tr>
<tr>
<td>( \cap )</td>
<td>intersection</td>
<td>( A \cap B )</td>
<td>( A ) intersect ( B )</td>
</tr>
<tr>
<td>( - )</td>
<td>difference</td>
<td>( A - B )</td>
<td>difference of ( A ) and ( B )</td>
</tr>
<tr>
<td>( \overline{A} )</td>
<td>complement</td>
<td></td>
<td>complement of ( A )</td>
</tr>
<tr>
<td>(</td>
<td>A</td>
<td>)</td>
<td>cardinality</td>
</tr>
</tbody>
</table>
1.5 Definitions

1) Subset: A is a subset of B if and only if every element of A is an element of B.

\[ \forall x \in A, \ x \in B \]

Notes:

a) There is one special subset, the empty set, noted \( \emptyset \) that belongs to all sets.

b) A is a subset of A: \( A \subseteq A \)

B) Union: The union of the sets A and B, denoted \( A \cup B \), is the set that contains those elements that are either in A, or in B, or in both.

\[ A \cup B = \{ x / x \in A \lor x \in B \} \]

C) Intersection: The intersection of the sets A and B, denoted \( A \cap B \), is the set that contains those elements that are both in A and B.

\[ A \cap B = \{ x / x \in A \land x \in B \} \]

D) Difference: The difference of the sets A and B, denoted \( A - B \) or \( A \setminus B \), is the set that contains those elements that are in A, but not in B.

\[ A - B = \{ x / x \in A \land x \notin B \} \]
E) Complement

Let \( A = \{1, 2, 3\} \). What would be the complement of \( A \)? Elements that do not belong to \( A \)? Then you would belong to this complement!

We need a robust definition.

First, we need to define the **domain**, a universal set, or universe, \( D \).

Then \( \overline{A} = \{ x \mid x \in D \land x \notin A \} \)

F) Cardinality

If a set \( S \) is finite, the cardinality of \( S \), denoted \( |S| \), is the number of elements of \( S \).

Addition principle: if \( A \) and \( B \) are two sets that are disjoint, i.e., \( A \cap B = \emptyset \) then

\[ |A \cup B| = |A| + |B| \]

Proof: obvious!
Inclusion-exclusion principle:

Let \( A \) and \( B \) be two sets in the same universe \( U \). Then

\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

Proof:

1) \( A \cup B = A \cup (B - A) \)

\[
A \cap (B - A) = \left\{ x \in U : x \in A \land (x \in B - A) \right\} = \left\{ x \in U : x \in A \cap (x \in B \land x \notin A) \right\} = \varnothing
\]

Therefore:

\[
|A \cup B| = |A| + |B - A| \quad (1)
\]

2) \( B = (B - A) \cup (A \cap B) \)

\[
(B - A) \cap (A \cap B) = \left\{ x \in U : (x \in B \land x \notin A) \land (x \in A \land x \in B) \right\} = \varnothing
\]

Therefore:

\[
|B| = |B - A| + |A \cap B| \quad (2)
\]

(1)-(2):

\[
|A \cup B| = |A| + |B| + |A \cap B|
\]
Set identities

A, B, and C sets in

Identity laws:

Denomination laws:

Idempotent

Double Complement

Distributive laws:

De Morgan's laws

Complement laws

A domain, a universe \( U \)

\[ A \cup \emptyset = A \]

\[ A \cap U = A \]

\[ A \cup U = U \]

\[ A \cap \emptyset = \emptyset \]

\[ A \cup A = A \]

\[ A \cap A = A \]

\[ \overline{A} = A \]

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

\[ A \cup B = \overline{A} \cap \overline{B} \]

\[ A \cap B = \overline{A} \cup \overline{B} \]

\[ A \cup A = U \]

\[ A \cap \overline{A} = \emptyset \]
Two examples of proofs

1) De Morgan's law: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

We show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ (1)

and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ (2)

1. Let $x \in \overline{A \cap B} : x \notin A \cap B$

   $\neg (x \in A \land x \in B)$

   $\neg (x \in A) \lor \neg (x \in B)$

   $x \in \overline{A} \lor x \in \overline{B}$

   $x \in \overline{A} \cup \overline{B}$

   Definition of complement
   and intersection

2. Let $x \in \overline{A} \cup \overline{B}$

   $x \in \overline{A} \lor x \in \overline{B}$

   $\neg x \in A \lor \neg x \in B$

   $\neg (x \in A \land x \in B)$

   $\neg (x \in A \cap B)$

   $x \in \overline{A \cap B}$

   Definition of union
   Definition of complement
   De Morgan's law
   Definition of intersection
   Definition of complement

Alternate proof using table of truth:

Let $A$ be a set in a universe $U$. We write "1" for $x \in A$, and "0" if $x \in U - A$
We can then generate the table of "truth": (9)

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>A</th>
<th>\bar{A}</th>
<th>\bar{B}</th>
<th>A \land B</th>
<th>\bar{A} \land \bar{B}</th>
<th>\bar{A} \lor \bar{B}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
</tr>
</tbody>
</table>

Therefore we have

\[ \bar{A} \lor \bar{B} = \bar{A} \lor \bar{B} \]

2) Distributivity

Show that

\[ A \lor (B \land C) = (A \lor B) \land (A \lor C) \]

We show that

1) \[ A \lor (B \land C) \subseteq (A \lor B) \land (A \lor C) \]

2) \[ (A \lor B) \land (A \lor C) \subseteq A \lor (B \land C) \]

1) Let \[ x \in A \lor (B \land C) \]

\[ x \in A \lor (x \in B \land x \in C) \]

\[ x \in A \lor (x \in B) \land (x \in C) \]

\[ (x \in A \lor x \in B) \land (x \in A \lor x \in C) \]

\[ x \in (A \lor B) \land (x \in A \lor x \in C) \]

\[ x \in (A \lor B) \land (A \lor C) \]

2) Let \[ x \in (A \lor B) \land (A \lor C) \]

\[ x \in (A \lor B) \land (A \lor C) \]

\[ (x \in A \lor x \in B) \land (x \in A \lor x \in C) \]

\[ (x \in A \lor x \in B) \land (x \in A \lor x \in C) \]

\[ x \in (A \lor B) \land (x \in A \lor x \in C) \]

\[ x \in (A \lor B) \land (A \lor C) \]

\[ x \in (A \lor B) \land (A \lor C) \]
\[ x \in (A \cup B) \land x \in (A \cup C) \]
\[ (x \in A \lor x \in B) \land (x \in A \lor x \in C) \]
\[ x \in A \lor x \in (B \cap C) \]
\[ x \in A \cup (B \cap C) \]

**Proof using truth table:**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>B \cap C</th>
<th>A \cup (B \cap C)</th>
<th>A \cup B</th>
<th>A \cup C</th>
<th>(A \cup B) \cap (A \cup C)</th>
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Therefore, \( A \cup (B \cap C) = (A \cup B) \land (A \cup C) \).
Example of application

Show that \( \overline{A \cup (B \cap C)} = (\overline{C \cup B}) \cap \overline{A} \)

\[
A \cup (B \cap C) = A \cap \overline{B} \cap \overline{C} \\
= A \cap (B \cup \overline{C}) \\
= A \cap (\overline{C} \cup B) \\
= (C \cup B) \cap \overline{A}
\]

De Morgan's law
\( \sqrt{\text{De Morgan's law}} \)
\( \sqrt{\text{Commutativity}} \)
\( \sqrt{\text{Commutativity}} \)

3. Generalized sets

3.1 Generalized union

The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

\[
A_1 \cup A_2 \ldots \cup A_n = \bigcup_{i=1}^{n} A_i
\]

3.2 Generalized intersection

The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection.

\[
A_1 \cap A_2 \ldots \cap A_n = \bigcap_{i=1}^{n} A_i
\]
Symbolic representation of generalized set:

\[
\bigcup_{i=1}^{N} A_i = \{ x \mid \exists i \in [1, \ldots, N], x \in A_i \}
\]

\[
\bigcap_{i=1}^{N} A_i = \{ x \mid \forall i \in [1, \ldots, N], x \in A_i \}
\]

**Negation**

\[\neg\]

\[x \notin \bigcup_{i=1}^{N} A_i \text{ means } \forall i \in [1, \ldots, N], x \notin A_i\]

\[x \notin \bigcap_{i=1}^{N} A_i \text{ means } \exists i \in [1, \ldots, N], x \notin A_i\]

3.3. The power set

Given a set \( S \), the power set of \( S \), \( \mathcal{P}(S) \) is the set of all subsets of \( S \).

**Example**: What is the power set of \( S = \{a, b, c\} \)?

\( \mathcal{P}(S) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \} \)
3.4 Cartesian product

Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$, denoted $A \times B$, is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

4. Computer representation of sets

There are various ways to represent sets using a computer. One method is to store the elements in an unordered list, or array. Operations on such lists however can be time consuming.

Bit string representation of a set

Let $D$ be the finite domain under consideration, whose elements are $D = \{a_1, \ldots, a_N\}$. A subset $A$ of $D$ is represented by the bit string of length $N$, such that the $i$th bit in this string is 1 if $a_i$ belongs to $A$, and 0 otherwise.
Example:

Let \( D = \{ a, b, c, d, e, f, g, h, i \} \)

Let \( A \) be the subset of the vowels contained in \( D \). The bit string representation of \( A \) is:

\[ S_A = 100010001 \]

Let \( B \) be the subset of the consonants contained in \( D \). \( B = \overline{A} \) and

\[ S_B = 011101110 \]