Functions

1) Some definitions about functions

**Definition 1**

Let $A$ and $B$ be sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$.

$A$ is the domain of $f$ and $B$ is the codomain.

**Definition 2**

A function $f$ from $A$ to $B$ is said to be one-to-one, or injective, if and only if $f(x) = f(y)$ implies $x = y$.

\[
\forall (x, y) \in A \times A, \quad f(x) = f(y) \Rightarrow x = y \quad (1)
\]

or

\[
\forall (x, y) \in A \times A, \quad x \neq y \Rightarrow f(x) \neq f(y) \quad (2)
\]

Examples: let us define the two functions $f$ and $g$ from $\mathbb{R}$ to $\mathbb{R}$ as:

\[
f(x) = 2x + 1 \quad \forall x \in \mathbb{R}
\]

\[
g(x) = x^2 \quad \forall x \in \mathbb{R}
\]

Are $f$ and $g$ one to one?
Let us show that \( f \) is one to one.

Let \( x_1, x_2 \) be two real numbers such that

\[
\begin{align*}
2x_1 + 1 &= 2x_2 + 1 \\
2x_1 &= 2x_2 \\
x_1 &= x_2
\end{align*}
\]

Therefore \( f \) is one to one.

Is \( g \) one to one?

\[
g(2) = 4 \quad \text{and} \quad g(-2) = 4 \quad \text{and} \quad 2 \neq -2
\]

We have shown there is a counter-example.

Therefore \( g \) is not one to one.

**Definition**

A function \( f \) from \( A \) to \( B \) is said to be **onto**, or **surjective**, if and only if, for every element \( b \) of \( B \), there is an element \( a \) of \( A \), with

\[
f(a) = b
\]

\[
\forall y \in B, \exists a \in A, \ f(a) = y
\]

**Example:** Show that the function \( f \) defined in the example above is \( f \) onto.
Let \( y \) be a real number.

The equation \( 2x + 1 = y \) has one solution:
\[
x_1 = \frac{1}{2} (y - 1)
\]

Therefore \( f(x_1) = 2x_1 + 1 = y \).

\( f \) is onto.

**Definition:** A function \( f \) from \( A \) to \( B \) is said to be a one-to-one correspondence, a bijection, if it is both injective and surjective.

**Example:**

Let \( f \) be a bijection from the set \( A \) to the set \( B \). The inverse function of \( f \), \( f^{-1} \), if the function that assigns an element of \( B \) the element of \( A \) such that \( f(a) = b \).

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is injective and surjective: it is bijective.

\( f^{-1} : \mathbb{R} \rightarrow \mathbb{R} \) is defined as:
\[
x \rightarrow \frac{1}{2} (x - 1)
\]
Composition of two functions:

Definition: Let \( g \) be a function from the set \( A \) to the set \( B \), and let \( f \) be a function from the set \( B \) to the set \( C \). The composition of the functions \( f \) and \( g \), denoted \( f \circ g \), is defined by:

\[
(f \circ g): A \rightarrow C, \quad a \rightarrow f[g(a)]
\]

Example:

Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) \( x \rightarrow 2x+1 \) and \( g: \mathbb{R} \rightarrow \mathbb{R} \) \( x \rightarrow x^2 \).

The function \( f \circ g \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \) such that:

\[
f \circ g (x) = f[g(x)] = f(x^2) = 2x^2 + 1
\]

for all \( x \) in \( \mathbb{R} \).

What about \( g \circ f \)? It is also a function from \( \mathbb{R} \) to \( \mathbb{R} \), defined by:

\[
g \circ f (x) = g[f(x)] = g(2x+1) = (2x+1)^2 = 4x^2 + 4x + 1
\]

Note: For most functions \( f \) and \( g \), \( f \circ g \neq g \circ f \).
2) **functions and cardinality of sets**

**Property:** We say that two sets $A$ and $B$ have the same cardinality if there exists a bijection from $A$ to $B$.

**Example**

$A = \mathbb{Z}$  \hspace{1cm} $B = IE$ (even integers)

Let $f: A \rightarrow B$

$x \rightarrow 2x$

$f$ is injective:

Let $(x, y) \in \mathbb{Z}^2$ such that $f(x) = f(y)$. Then $2x = 2y$, therefor $x = y$.

$f$ is surjective:

Let $y \in IE$, $y$ is even therefore there exists $k \in \mathbb{Z}$ such that $y = 2k$ \hspace{1cm} $y = f(k)$

Therefore: \hspace{1cm} $|\mathbb{Z}| = |IE|$

**Some definitions**

- Any set $S$ with cardinality less than or the natural number, $\mathbb{N}$, $|S| < |\mathbb{N}|$ is said to be a finite set.

- Any set $S$ that has the same cardinality than $\mathbb{N}$, $|S| = |\mathbb{N}|$ is said to be a countably infinite set.
Any set with cardinality greater than \( |\mathbb{N}| \), is said to be uncountable. \( \mathbb{R} \) is uncountable. Show that \( \mathbb{Z} \) is countably infinite.

Let \( f: \mathbb{N} \to \mathbb{Z} \):

\[
\begin{array}{c|cccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 f(n) & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 \\
\end{array}
\]

\( f \) is bijective. Therefore \( |\mathbb{Z}| = |\mathbb{N}| \).

3) Floor and ceiling functions

**Floor function**  
The floor function assigns to the real number \( x \) the largest integer that is less than or equal to \( x \).

\[
\lfloor x \rfloor: \mathbb{R} \to \mathbb{Z} \\
x \mapsto \lfloor x \rfloor \text{ such that } \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1
\]

**Examples:**

\[
\lfloor n \rfloor = n \quad \forall n \in \mathbb{N}
\]

\[
\lfloor 2.1 \rfloor = 2
\]

\[
\lfloor -3.5 \rfloor = -4
\]
3. Ceiling function

The ceiling function assigns to the real number \( x \) the smallest integer that is greater than or equal to \( x \).

\[ \Gamma I : \mathbb{R} \rightarrow \mathbb{Z} \]

\[ x \rightarrow \lceil x \rceil \text{ such that } \lceil x \rceil - 1 < x \leq \lceil x \rceil \]

3. Useful properties of the floor and ceiling functions

a) \( x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1 \)

b) \( -x \lfloor x \rfloor = -\lceil x \rceil \)

\( \lceil -x \rceil = -\lfloor x \rfloor \)

c) \( \lfloor x + n \rfloor = \lfloor x \rfloor + n \)

\( \lceil x + n \rceil = \lceil x \rceil + n \)

3.4) Examples

Example 1: Prove that \( \forall x \in \mathbb{R}, \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor \)

Proof: Let \( x \) be a real number. We set \( \text{LHS} = \lfloor 2x \rfloor \) and \( \text{RHS} = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor \). There exists \( m \in \mathbb{Z} \) and \( \varepsilon \in \mathbb{R}, 0 \leq \varepsilon < 1 \) such that \( x = m + \varepsilon \). By definition of the floor function, \( \lfloor x \rfloor = m \).
\[ \varepsilon \text{ is called the fractional part of } x \]

What about \(2x\)?

\[ 2x = 2n + 2\varepsilon, \quad \text{with } 0 \leq \varepsilon < 2 \]

**Proof by case:**

\[ i) \quad 0 \leq \varepsilon < \frac{1}{2} \]

Then \( 0 \leq 2\varepsilon < 1 \)

Therefore \( 2n \leq 2n + 2\varepsilon < 2n + 1 \) i.e. \( LHS = 2n \)

Similarly:

\[ 0 \leq \varepsilon < \frac{1}{2} \]

\[ \frac{m + 1}{2} \leq m + \varepsilon + \frac{1}{2} < m + 1 \]

Since \( m < m + \frac{1}{2} \) and \( x = m + \varepsilon \), we get:

\[ m < x + \frac{1}{2} < m + 1 \]

Therefore \( Lx + \frac{1}{2} \mid = m \)

We have:

\[ \text{LHS} = L2x \mid = 2n \]

\[ \text{RHS} = Lx \mid + Lx + \frac{1}{2} \mid = m + m = 2n \]

The property is true.

\[ ii) \quad \frac{1}{2} \leq \varepsilon < 1 \]

Then \( 1 \leq 2\varepsilon < 2 \)

Therefore \( 2n + 1 \leq 2n + 2\varepsilon < 2n + 2 \) and \( LHS = L2x \mid = 2n + 1 \)

Similarly:
\[
\begin{align*}
\frac{1}{2} & \leq \varepsilon < 1 \\
\frac{1}{2} m & \leq m + \varepsilon + \frac{1}{2} < m + \frac{3}{2} < m + 2 \\
\text{Therefore:} & \quad Lx + \frac{1}{2} \leq m + 1 \\
\text{And} & \quad \text{RHS} = Lx + \frac{1}{2} \leq 2m + 1 \\
\text{The property is true.} \\
\text{Conclusion:} & \quad \text{In all cases, we have LHS = RHS}
\end{align*}
\]

**Example 2**: Prove that \( \forall x \geq 0, x \in \mathbb{R}, L \sqrt{|x|} = L \sqrt{x} \)

**Proof**: Let \( x \) be a positive real number.

Let \( m = L \sqrt{|x|} \) and let \( k = Lx \).

By definition of \( L \sqrt{|x|} \):

\[0 \leq m \leq \sqrt{x} \leq m + 1\]

Therefore:

\[m^2 \leq x < (m+1)^2 \quad (1)\]

By definition of \( Lx \):

\[k \leq Lx \leq k + 1 \quad (2)\]

From (2):

\[k < (m+1)^2\]

From (1): \( m^2 \leq x \) and \( k \) is the largest integer smaller than \( x \).

Therefore:

\[m^2 \leq k < (m+1)^2\]

Therefore:

\[L \sqrt{k} = m, \quad \text{and} \quad L \sqrt{|x|} = L \sqrt{x}\]
4. Growth of Functions

4.1 Definition: Let \( f \) and \( g \) be two functions from \( \mathbb{R} \) or \( \mathbb{N} \) to \( \mathbb{R} \). We say that \( f(x) \) is \( O(g(x)) \) (big "O") if there exists two constants \( C \) and \( k \) such that
\[
\forall x > k, \quad |f(x)| \leq C |g(x)|
\]

In symbols:
\[
\exists C \in \mathbb{R}^+, \forall x \in \mathbb{R}, \quad \forall x > k, \quad |f(x)| \leq C |g(x)|
\]

Example: Show that \( f(x) = x^2 + 2x + 1 \) is \( O(x^2) \).

We note that:
\[
\begin{align*}
\text{if } x > 1, \quad x < x^2 \\
\text{therefore, } \quad 2x < 2x^2 \\
\text{We also know, } \quad 1 < x^2 \\
\text{and } \quad x^2 \leq x^2
\end{align*}
\]

Therefore:
\[
x^2 + 2x + 1 \leq 4x^2
\]

We choose \( k = 1 \) and \( C = 4 \)

Therefore \( f(x) \) is \( O(x^2) \)
4.2. Important theorems

**Theorem 1**
Let \( f(x) = a_n x^n + \ldots + a_1 x + a_0 \)
where \( x \) and \( a_i \) are real numbers. Then
\[
|f(x)| = O(x^n)
\]

Proof: let \( x \) be a real number.

If \( x > 1 \) then \( x^i < x^n \) for \( 0 \leq i < n - 1 \)

\[
|f(x)| \leq |a_n x^n + \ldots + a_0| \\
\leq |a_n| x^n + \ldots + |a_0| x^n \\
\leq (|a_n| + \ldots + |a_0|) x^n
\]

We choose \( k = 1 \) and \( C = |a_n| + \ldots + |a_0| \)

**Theorem 2**
Let \( f : \mathbb{N} \to \mathbb{R} \)

\[
f(n) = O(n^2)
\]

**Theorem 3**
Let \( f : \mathbb{N} \to \mathbb{R} \)

\[
f(n) \to n!
\]

Then \( f(n) = O(n^m) \)

**Theorem 4**
Suppose that \( f_1(x) = O(g_1(x)) \)
and \( f_2(x) = O(g_2(x)) \). Then

\( f_1 + f_2 \) also is \( O \) of \( \max(g_1(x), g_2(x)) \).
**Theorem 5:** Suppose that \( f_1 (x) \) is \( O(q_1(x)) \) \( (12) \)
and \( f_2 (x) \) is \( O(q_2(x)) \). Then
\[
f_1 f_2 (x) \) is \( O\left( q_1(x) q_2(x) \right) \)

**Examples:** Find a big \( O \) estimate for:

- \( f(x) = 2x^3 + x^2 \log x \)
  
  \[ 2x^3 \text{ is } O(x^3) \]
  
  \( \forall x > 1 \), \( x^2 \log x < x^3 \) therefore \( x^2 \log x \) is \( O(x^3) \)
  
  Therefore \( f(x) \) is \( O(x^3) \)

- \( f(x) = \frac{x^4 + x^2 + 1}{x^3 + 1} \)

  For \( x > 1 \), \( \sqrt{f(x)} = x \frac{1 + \frac{1}{x^2} + \frac{1}{x^4}}{1 + \frac{1}{x^3}} \leq 3x \)
  
  Therefore \( \sqrt{f(x)} \) is \( O(x) \)

- More generally:
  
  \[ f(x) = a_n x^n + \cdots + a_0 \text{ is } O\left( \left| \frac{a_n}{b_p} \right| x^{n-p} \right) \]

4.3. Extending Big-O
**Definition**

Let \( f \) and \( g \) be functions from \( \mathbb{Z} \) or \( \mathbb{R} \) to \( \mathbb{R} \). We say that \( f(x) \) is \( \Omega(g(x)) \) if there exist constants \( C \) and \( k \) such that:

\[
|f(x)| > C |g(x)| \quad \forall x > k, \quad \exists C \in \mathbb{R}^+, \forall f(x) \geq C |g(x)|
\]

\( f(x) \) is "big Omega" of \( g(x) \)

**Definition**

Let \( f \) and \( g \) be functions from \( \mathbb{Z} \) or \( \mathbb{R} \) to \( \mathbb{R} \). We say that \( f(x) \) is \( O(g(x)) \) if \( f(x) \) is \( O(x) \) and \( f(x) \) is \( \Omega(g(x)) \)

\( f(x) \) is "big Theta" of \( g(x) \), or

\( f(x) \) is \( \Theta(g(x)) \)

\[
1 + \ldots + m = \frac{m(m+1)}{2} \geq m^2
\]

Therefore \( 1 + \ldots + m \) is \( \Omega(m^2) \)

We know that \( 1 + \ldots + m \) is \( O(m^2) \)

Therefore \( 1 + \ldots + m \) is \( \Theta(m^2) \)