Logic (mathematical language)

1) Propositions

Definition: A proposition, or statement, is a declarative sentence that is either true (T) or false (F).

Examples:

Sentence: Proposition or not | Truth value
---|---
1 + 1 = 2 | Yes | True
1 + 4 = 3 | Yes | True
Today is Sunday | Yes | False
x + 3 = 5 | No | Don't know

I am lying now

No, but...

A proposition, T or F

Table of truth:

<table>
<thead>
<tr>
<th>P</th>
<th>T</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Compound proposition

1) Negation: Let \( p \) be a proposition. The sentence "It is not the case that \( p \)" is another proposition, called the negation of \( p \), denoted \( \neg p \). It is read "not \( p \)".

Truth table

\[
\begin{array}{c|c}
  p & \neg p \\
  \hline
  T & F \\
  F & T \\
\end{array}
\]

2) Conjunction: The conjunction of two propositions \( p \) and \( q \) is the compound proposition that we write \( p \land q \), and read "\( p \) and \( q \)" which is only true if \( p \) is true and \( q \) is true.

Table of truth:

\[
\begin{array}{ccc}
  p & q & p \land q \\
  \hline
  T & T & T \\
  T & F & F \\
  F & T & F \\
  F & F & F \\
\end{array}
\]
**Disjunction**: The disjunction of 2 propositions $p$ and $q$ is the compound proposition written as $p \lor q$, and read "$p$ or $q$" that is true if either $p$ is true, or $q$ is true, or both are true.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
<th>$p \lor q$</th>
<th>$p \oplus q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
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<tr>
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**Exclusive or**: $p \oplus q$

**Logical equivalence**: Two propositions $p$ and $q$ are logically equivalent if they always have the same truth values. Notation: we write $p \equiv q$

**Statement for 2 propositions $p$ and $q$**

$$\neg (p \lor q) \equiv \neg p \lor \neg q$$
Truth table:

<table>
<thead>
<tr>
<th>P</th>
<th>q</th>
<th>Tp</th>
<th>Tq</th>
<th>PAq</th>
<th>(7PAq)</th>
<th>7Pv7q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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Show that \( \neg (p \lor q) \implies \neg p \land \neg q \)

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<th>P</th>
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<td>T</td>
</tr>
</tbody>
</table>
Table of truth:

\[
\begin{array}{cccccc}
P & \neg P & P \land q & P \lor q & P \oplus q \\
T & T & F & T & T \\
T & F & F & T & T \\
F & T & F & T & T \\
F & F & T & T & F \\
\end{array}
\]

\[P \oplus q \equiv (P \lor q) \land (\neg (P \land q))\]

De Morgan's laws:

\[
\begin{align*}
\neg (P \land q) & \equiv \neg P \lor \neg q \\
\neg (P \lor q) & \equiv \neg P \land \neg q
\end{align*}
\]

\[P \land \neg P \equiv F\]

A contradiction is a statement that is always false.

\[P \lor \neg P \equiv T\]

A tautology is a statement that is always true.
Show that \((pq) \lor (\neg p \land \neg q)\) is a tautology.

Proof by logical equivalence:

\[
(pq) \lor (\neg p \land \neg q) \equiv (pq) \lor (\neg (pq)) \quad \text{De Morgan's law}
\]

Let us define \(A = pq\)

Then

\[
(pq) \lor (\neg p \land \neg q) \equiv A \lor \neg A \equiv T
\]

\[
a \ast (b + c) = a \ast b + a \ast c
\]

\[
P \land (pq) \equiv (P \land q) \lor (P \land r)
\]

\[
P \lor (q \land r) \equiv (pq) \lor (pr)
\]

\[
P \lor (q \land r) \equiv (pq) \land (pr)
\]
You arrive on Smaillyan's island and you meet Bill and Sally.

Bill: "The two of us are knights."

Sally: "Bill is a knave.

<table>
<thead>
<tr>
<th>Knight</th>
<th>Knight</th>
<th>0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knight</td>
<td>Knave</td>
<td>1</td>
<td>F</td>
</tr>
<tr>
<td>Knave</td>
<td>Knight</td>
<td>2</td>
<td>T</td>
</tr>
<tr>
<td>Knave</td>
<td>Knave</td>
<td>3</td>
<td>F</td>
</tr>
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</table>

0: Sally is a knight that would lie.

1: Bill is a knight that would lie.

2: Sally is a knave that would tell the truth.
Bill: I am a knave or Sally is a knight.
Sally: I am a knight.

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<th>Sally</th>
<th>PB</th>
<th>Ps</th>
</tr>
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<tbody>
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<td>Knight</td>
<td>Knight</td>
<td>T</td>
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Bill: We are the same kind
Sally: We are of different kind.

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