Exercise 1

Let $p$ and $q$ be two propositions. The proposition $p \text{ NOR } q$ is true when both $p$ and $q$ are false, and it is false otherwise. It is denoted $p \downarrow q$.

a) Write down the truth table for $p \downarrow q$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \downarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

b) Show that $p \downarrow q$ is equivalent to $\neg(p \lor q)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \downarrow q$</th>
<th>$p \lor q$</th>
<th>$\neg(p \lor q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
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<td>T</td>
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</table>

Therefore $p \downarrow q$ is equivalent to $\neg(p \lor q)$

c) Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator $\downarrow$.
Exercise 2

Let \( P(x) \) be the statement “\( x = x^2 \)”. If the domain consists of the integers, what are the truth values of the following statements:

a) \( P(0) \)
   \( P(0): 0 = 0^2: \text{true} \)

b) \( P(1) \)
   \( P(1): 1 = 1^2: \text{true} \)

c) \( P(2) \)
   \( P(2): 2 = 2^2: \text{false} \)

d) \( P(-1) \)
   \( P(-1): -1 = (-1)^2: \text{false} \)

e) \( \exists x \; P(x) \)
   The statement is true: \( P(1) \) is true: proof by example

f) \( \forall x \; P(x) \)
   The statement is false: \( P(2) \) is false: proof by counter-example

Exercise 3

Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase “It is not the case that.”)

a) All dogs have fleas.
   \( \forall d \in \text{Dogs}, d \text{ has fleas.} \)
   Negation: There exists a dog that does not have flea.

b) There exists a horse that can add.
   \( \exists h \in \text{Horses}, h \text{ can count.} \)
   Negation: All horses cannot add.
c) Every koala can climb.
\[ \forall k \in \text{Koalas}, k \text{ can climb.} \]
Negation: There exists koala that cannot climb.

d) No monkey can speak French.
\[ \forall m \in \text{Monkeys}, k \text{ m cannot speak French.} \]
Negation: There is a monkey that can speak French

e) There exists a pig that can swim and catch fish.
\[ \exists p \in \text{Pigs}, p \text{ can swim and catch fish.} \]
Negation: Every pig either cannot swim, or cannot catch fish.

Exercise 4

a) Let \( a \) and \( b \) be two integers. Prove that if \( n = ab \), then \( a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \)
We use a proof by contradiction. Let us suppose that \( a > \sqrt{n} \) and \( b > \sqrt{n} \). Then \( ab > n \), i.e. \( n > n \). We have reached a contradiction. Therefore the property is true.

b) Prove or disprove that there exists \( x \) rational and \( y \) irrational such that \( x^y \) is irrational.
Let \( x = 2 \) and \( y = \sqrt{2} \). Then \( x^y = 2^{\sqrt{2}} \). There are two cases:
- \( 2^\sqrt{2} \) is irrational. We are done
- \( 2^\sqrt{2} \) is rational. Let us define then \( x = 2^{\sqrt{2}} \) and \( y = \sqrt{2}/4 \). Then

\[
x^y = \left(2^{\sqrt{2}}\right)^{\sqrt{2}/4} = 2^{\sqrt{2}/4} = 2^{1/2} = \sqrt{2}
\]

i.e. \( x^y \) is irrational.

We have shown that there exists \( x \) rational and \( y \) irrational such that \( x^y \) is irrational but we do not know the values of \( x \) and \( y \): non-constructive proof.

c) Show that \( \sqrt{2} \) is irrational.
We do a proof by contradiction. Let us suppose that there exists two integers \( a \) and \( b \), with \( b \neq 0 \), such that \( \sqrt{2} = \frac{a}{b} \), with \( \frac{a}{b} \) reduced, i.e. \( a \) and \( b \) do not have common factors.

Then: \( b \sqrt{2} = a \), i.e. \( 2b^3 = a^3 \). Therefore \( a^3 \) is even... it is easy to show that then \( a \) is even.

Since \( a \) is even, there exists an integer \( k \) such that \( a = 2k \). Therefore \( 2b^3 = 8k^3 \), i.e. \( b^3 = 4k^3 \), hence \( b^3 \) is even, and \( b \) is even.

We have reached a contradiction: \( \frac{a}{b} \) reduced \( \rightarrow \) \( a \) is even and \( b \) is even. Therefore \( \sqrt{2} \) is irrational.
d) There exists no integers $a$ and $b$ such that $21a + 30b = 1$.

We do a proof by contradiction. Let us suppose that there exists two integers $a$ and $b$ such that $21a + 30b = 1$. Then $3(7a + 10b) = 1$. Since $7a + 10b$ is an integer, 1 would be a multiple of 3; we have reached a contradiction. Therefore the property is true.